April 1961 · Vol. 15, No. 74

Mathematics of Computation

A journal devoted to advances in numerical analysis, the application of computational methods, mathematical tables, high-speed calculators and other aids to computation

Formerly: Mathematical Tables and other Aids to Computation

Published Quarterly by the
National Academy of Sciences—National Research Council

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Mt. Royal and Guilford Avenues, Baltimore 2, Md.
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Polynomial Approximations to Finitely Oscillating Functions

By William J. Kammerer

1. Introduction. Chandler Davis [1] established the following theorem: *If $v_0, v_1, \cdots, v_n$ are real numbers such that $v_0 > v_1, v_1 < v_2, v_2 > v_3, \cdots$, then there exists a unique polynomial P of degree n and a set of points $y_0, y_1, \cdots, y_n$ such that*

$$P(y_i) = v_i \qquad i = 0, 1, \cdots, n \tag{1}$$

$$P'(y_i) = 0 \qquad i = 1, 2, \cdots, n - 1 \tag{2}$$

$$0 = y_0 < y_1 < \cdots < y_n = 1.$$

The main result of this paper is an algorithm for the calculation of this polynomial, which is first motivated by an independent proof for the existence of P.

A function f is said to be finitely oscillating if it has at most a finite number of relative extrema. The following mode of approximating a continuous finitely oscillating function f in the uniform norm so that the oscillations are preserved, is discussed: first obtain a polynomial P of minimal degree which has the same variation as f and then obtain an increasing polynomial Q such that $P(Q)$ agrees with f at all its relative extrema. Two theorems are given in the last section, to show that this method of approximation is always possible.

2. Proof of Theorem. Let D and D_0 denote the set of all $n + 1$ tuples $X = (x_0, x_1, \cdots, x_n)$ such that $0 = x_0 < x_1 < \cdots < x_n = 1$ and $0 = x_0 < x_1 < \cdots < x_n \leqq 1$ respectively. Let $\mathcal{P}$ denote the class of all polynomials p of degree n, for which there exists an element $X \varepsilon D$, such that $p(x_i) = v_i (i = 1, 2, \cdots, n)$. A polynomial $p \varepsilon \mathcal{P}$ can be written in the following form:

$$\begin{aligned} p(x) &= [x_0] + [x_0, x_1](x - x_0) \\ &\quad + \cdots + [x_0, x_1, \cdots, x_n](x - x_0) \cdots (x - x_{n-1}) \end{aligned} \tag{3}$$

where the bracket function is defined by

$$\begin{gathered} [x_i] = v_i \\ [x_i, x_{i+1}, \cdots, x_{i+k}] = \frac{[x_{i+1}, \cdots, x_{i+k}] - [x_i, \cdots, x_{i+k-1}]}{x_{i+k} - x_i}. \end{gathered} \tag{4}$$

Lemma 2.1 *If $X \varepsilon D_0$ and $n \geqq 1$ then $[X] > 0$ for n even and $[X] < 0$ for n odd.*

Proof. The proof proceeds by induction. The lemma can be shown to hold for $n = 1$ by direct computation. Assuming it true for $n = k - 1$, one has $[x_0, x_1, \cdots, x_{k-1}] < 0$ (>0) and $[x_1, x_2, \cdots, x_k] > 0$ (<0) for k even (odd). The lemma is therefore true for $n = k$ by (4).

Received March 11, 1960. This work was supported in part by the Office of Ordnance Research, U. S. Army.

LEMMA 2.2. a) *If n is even (odd), the function $[X]$ assumes its minimum (maximum) in D.*

b) *Let $|[X]|$ attain its minimum value for $X \varepsilon D$, at $Y = (y_0, y_1, \cdots, y_n)$. Then the polynomial $p(y_k) = v_k (k = 0, 1, \cdots, n)$ is the desired polynomial P.*

Proof. By lemma 2.1, $[X]$ is of constant sign in D. The function $|[X]|$ approaches infinity as X approaches the boundary of D, and is continuous in the compact set $x_0 + \epsilon \leqq x_1, x_1 + \epsilon \leqq x_2, \cdots, x_{n-1} + \epsilon \leqq x_n$ for arbitrarily small $\epsilon > 0$.

The following notation is introduced, to simplify the proof of part b:

$$\begin{aligned} \alpha(p, i) &= \min \{x : x \varepsilon I(p, i)\} \\ \beta(p, i) &= \max \{x : x \varepsilon I(p, i)\} \end{aligned} \qquad i = 0, 1, \cdots, n$$

where

$$I(p, i) = \left\{x : \begin{bmatrix} p(x) \geqq v_i \text{ if } v_i > v_{i+1} \\ p(x) \leqq v_i \text{ if } v_i < v_{i+1} \end{bmatrix} \text{ and } x_{i-1} < x < x_{i+1}\right\}$$

$$i = 1, 2, \cdots, n-1 \qquad I(p, 0) = 0 \qquad I(p, n) = 1 \qquad \text{and} \quad p \varepsilon \mathcal{P}.$$

Let Y be a vector in D at which $|[X]|$ attains its minimum and let p be the polynomial in $\mathcal{P}$ which satisfies $p(y_i) = v_i$. If p does not satisfy (2), then there exists an integer k, such that $\alpha(p, k) \neq \beta(p, k)$. Let $x_k = \frac{1}{2}\{\alpha(p, k) + \beta(p, k)\}$ and consider the following two polynomials

$$\begin{aligned} p(x) &= h(x) + [Y]g_k(x, Y) \\ p_1(x) &= h(x) + [y_0, \cdots y_{k-1}, x_k, y_{k+1}, \cdots, y_n]g_k(x, Y) \end{aligned}$$

where $h(x)$ is a polynomial of degree $n - 1$, and

$$g_k(x, Y)(x - y_k) = \prod_{i=0}^{n} (x - y_i). \tag{5}$$

By construction one has $|p_1(x_k)| < |p(x_k)|$. Investigation of the possible cases contradicts the hypothesis that Y is a relative extrema of $[X]$.

It should be observed that the above theorem and proof are valid for polynomials of the form $\sum_{k=0}^{n} a_k f(x)^k$ where f is a strictly increasing differentiable function on $[0, 1]$. The proof is identical except for notation.

3. An Iterative Procedure.

Step 1. Choose an arbitrary element $X_1 = (x_0^1, x_1^1, \cdots, x_n^1)$ in D.

Step 2. With one of the standard interpolating formulas construct the polynomial $p_1 \varepsilon \mathcal{P}$, such that $p_1(x_k^1) = v_k$, $k = 0, 1, \cdots, n$.

Step 3. Determine the vector $X_2 = (x_0^2, x_1^2, \cdots, x_n^2)$ in D such that $p_1'(x_i^2) = 0$ for $i = 1, 2, \cdots, n - 1$.

We now have a new element $X_2 \varepsilon D$. To obtain p_2, repeat this process beginning with step 2, using X_2 in place of X_1 and making the obvious change in subscripts.

Continuing this procedure, we obtain a recursive process for obtaining a sequence $\{p_i\}$.

THEOREM 3.1. *The sequence* $\{p_i\}$ *converges uniformly to* P.

Proof. The proof will proceed by a series of lemmas.

LEMMA 3.2. *If* $p \varepsilon \mathcal{P}$, *then*

$$[x_0, \cdots, x_{k-1}, \alpha(p, k), x_{k+1}, \cdots, x_n] = [x_0, \cdots, x_{k-1}, \beta(p, k), x_{k+1}, \cdots, x_n]$$
$$k = 1, 2, \cdots, n-1.$$

Proof. By (3), p can be written in the following two forms

$$p(x) = h(x) + [x_0, \cdots, x_{k-1}, \alpha(p, k), x_{k+1}, \cdots, x_n]\, g_k(x, X) \tag{6}$$

$$p(x) = h(x) + [x_0, \cdots, x_{k-1}, \beta(p, k), x_{k+1}, \cdots, x_n]\, g_k(x, X) \tag{7}$$

where $h(x)$ is a polynomial of degree $n-1$ and $g_k(x, X)$ is defined as in (5). To obtain the desired result, subtract (6) from (7).

LEMMA 3.3. *Let* p *be an element of* $\mathcal{P}$ *and let* $N(p) = max_i \mid m_i - v_i \mid$ *where* $m_i = max \mid p(x) \mid$ *for* $x \varepsilon [\alpha(p, i), \beta(p, i)]$ *and* $i = 1, 2, \cdots, n-1$. *Then the following inequalities hold for the sequence* $\{p_i\}$:

a) *If* $N(p_i) \neq 0$ *then* $\mid [X_{i+1}] \mid < \mid [X_i] \mid$

b) $\mid p_i(x) - p_{i+1}(x) \mid \leqq \mid [X_i] - [X_{i+1}] \mid$ *for every* $x \varepsilon [0, 1]$

c) $N(p_i) \leqq max_{0 \leqq x \leqq 1} \mid p_i(x) - p_{i+1}(x) \mid$.

Proof. Let i be any positive integer and let $\{h_k\}$, $k = 1, 2, \cdots, n-1$ be the polynomials of degree n such that

$$h_k(x_m^{i+1}) = v_m \qquad \text{for } m = 0, 1, \cdots, k$$

$$h_k(\beta_m) = v_m \qquad \text{for } m = k+1, \cdots, n$$

where $\beta_m = \beta(p_i, m)$. Let Z_k denote the vector $(x_0^{i+1}, \cdots, x_k^{i+1}, \beta_{k+1}, \cdots, \beta_n)$. Then $I(h_k, m) \subset I(h_{k+1}, m)$ for $m = k+2, \cdots, n$, for otherwise $h_k(x) - h_{k+1}(x)$ would be identically equal to zero, instead of having $n-1$ simple roots in $[0, 1]$. As in the proof of lemma 2.2, we have $\mid [Z_0] \mid \geqq \mid [Z_1] \mid \geqq \cdots \geqq \mid [Z_n] \mid$ with at least one of the inequalities being strict, since by assumption, $N(p_i) \neq 0$. This proves part *a*.

For $k = 0, 1, \cdots, n-1$ one has $\mid h_k(x) - h_{k+1}(x) \mid = \mid [Z_k] - [Z_{k+1}] \mid \cdot \mid g_k(x, Z_k) \mid \leqq \mid [Z_k] - [Z_{k+1}] \mid$. Now apply the triangle law to obtain *b*.

The proof of part *c* follows from the existence of an integer k, such that $N(p_i) = \mid p_i(x_k^{i+1}) - p_{i+1}(x_k^{i+1}) \mid$.

To complete the proof of theorem 3.1, observe that lemmas 2.1 and 3.3.a imply that $\{[X_i]\}$ is a Cauchy sequence. Lemma 3.3.b implies $\{p_i\}$ also forms a Cauchy sequence in the uniform norm on $[0, 1]$ and therefore convergent to a polynomial P of degree n. Part *c* of lemma 3.3 implies that P satisfies conditions 1 and 2.

This iteration can be carried out by the use of standard subroutines available in most computer libraries, and from all empirical evidence the convergence seems quite rapid. To illustrate, we shall calculate the third-degree Chebyshev polynomial on $[0, 1]$ by the use of this method, starting with $X_1 = (0, .25, .5, 1)$.

$p_1(x) = 1 - 22x + 68x^2 - 48x^3$

$$X_2 = (0, .20723911, .73720533, 1)$$

$p_2(x) = 1 - 18.400541x + 48.970905x^2 - 32.570364x^3$

$$X_3 = (0, .24979357, .75256767, 1)$$

$p_3(x) = 1 - 18.000432x + 48.002138x^2 - 32.001707x^3$

$$X_4 = (0, .24999339, .74999783, 1)$$

$p_4(x) = 1 - 17.999999x + 47.999999x^2 - 31.999999x^3$

4. Approximation by Composition.

THEOREM 4.1. *Let $X = (x_0, x_1, \cdots, x_n)$, $Y = (y_0, y_1, \cdots, y_n)$ be any two elements in D. Then there exists a polynomial Q such that $Q'(x) \geqq 0$ on* [0, 1] *and $Q(x_i) = y_i$, $i = 0, 1, \cdots, n$.*

Proof. Define the elements Z_1 and Z_2 in D as follows:

$$Z_1 = (z_0^1, z_1^1, \cdots, z_n^1) = (y_0, \tfrac{1}{4}(y_2 - y_1) + y_1, \tfrac{1}{4}(y_3 - y_2) + y_2, \cdots, y_n)$$

$$Z_2 = (z_0^2, z_1^2, \cdots, z_n^2) = (y_0, \tfrac{1}{4}(y_1 - y_0) - y_1, \tfrac{1}{4}(y_2 - y_1) - y_2, \cdots, y_n).$$

Let f_i for $i = 1, 2, 3, \cdots, 2^{n-1}$ be distinct piecewise linear functions, which are linear on the intervals $[y_{i-1}, y_i]$ $i = 1, 2, \cdots, n$ and such that $f_i(x_k) = z_k^1$ or z_k^2, $k = 0, 1, \cdots, n$. Define $\epsilon = \min_i \min_k | z_i^k - y_i |$ for $i = 1, 2, \cdots, n - 1$, $k = 1, 2$. Using the fact that the Bernstein polynomials of a continuous increasing function are increasing and uniformly convergent, there exist increasing polynomials $Q_i(x)$ on [0, 1] such that $| f_i(x) - Q_i(x) | < \frac{1}{2}\epsilon$ for $i = 1, 2, \cdots, 2^{n-1}$ (see Lorentz [7] p. 20–23). The vector Y is contained in the convex hull of the vectors $(Q_i(x_0), Q_i(x_1), \cdots, Q_i(x_n))$, $i = 1, 2, \cdots, 2^{n-1}$, and therefore there exists a convex linear combination of the Q_i's which will give rise to a desired polynomial Q.

THEOREM 4.2. *Let f be a continuous finitely oscillating function on* [0, 1] *and let $\varepsilon > 0$ be given. Then there exist polynomials $P(y)$ and $Q(x)$, such that*

a) *$f(x)$ and $P(y)$ are equal at their corresponding relative extrema. At the relative extrema of f,*

$$P(Q) = f \quad \text{and} \quad \frac{dP(Q)}{dx} = 0.$$

b) *The polynomial Q is increasing and $| f(x) - P(Q(x)) | < \epsilon$ on* [0, 1].

Proof. Let the partition $X = (x_0, x_1, \cdots, x_n)$ be the points of the relative extrema of f on [0, 1]. By a previous theorem, there exists a polynomial $P(y)$ and a partition $Y = (y_0, y_1, \cdots, y_n)$ such that $P(y_i) = f(x_i)$, $i = 0, 1, \cdots, n$ and $P'(y_i) = 0$, $i = 1, 2, \cdots, n - 1$. Let X' be a refinement of the partition X which satisfies the following condition $x_i = x_{i1} < x_{i2} < \cdots < x_{iN_i} = x_{i+1}$ such that $N_i | f(x_{ij}) - f(x_{ij+1}) | = | f(x_i) - f(x_{i+1}) | \leqq N_i \epsilon$ for $i = 0, 1, \cdots, n - 1$, $j = 1, 2, \cdots, N_i$. Define Y' to be the refinement of the partition Y such that $y_{ij} \varepsilon [y_i, y_{i+1}]$ and $f(x_{ij}) = P(y_{ij})$ for $i = 0, 1, \cdots, n - 1, j = 1, 2, \cdots, N_i$. By theorem 4.1 there exists an increasing polynomial $Q(x)$ on [0, 1], such that $Q(x_{ij}) = y_{ij}$, and therefore by construction $| p(Q(x)) - f(x) | < \epsilon$ on [0, 1].

As a concluding remark, let C represent the class of all composite polynomials of the form $P(Q)$, where both P and Q are of degree greater than unity and $Q' \geqq 0$

on [0, 1]. Not every polynomial of degree four or larger can be so written (see H. Levi [6]). However, since every continuous function on [0, 1] can be uniformly approximated by a polynomial, (i.e., a finitely oscillating function) one finds that the completion of C in the uniform norm on [0, 1] is the set of all continuous functions on [0, 1].

The question of the existence of variation preserving approximations arose from the investigation of syntonic functions by Professor P. C. Hammer in [2] and [3]. I would also like to thank the referee for pointing out that the ideas in references [4], [5] and [8] bear a relation to the problem treated here.

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Table of a Weierstrass Continuous Non-Differentiable Function

By Herbert E. Salzer and Norman Levine

Many studies have been made of continuous non-differentiable functions [1], the most famous of which is Weierstrass's $W(a, b, x)$ defined by

$$(1)\qquad W(a, b, x) = \sum_{n=1}^{\infty} a^n \cos(b^n \pi x), \qquad 0 < a < 1,\ b \text{ an odd integer.}$$

It is shown in some books [1], [2] that for

$$(2)\qquad ab > 1 + \frac{3\pi}{2},$$

$W(a, b, x)$ is continuous everywhere and has no derivative anywhere, but Bromwich [3] improved this condition to

$$(3)\qquad ab > 1 + \frac{3\pi}{2}(1 - a),$$

which, according to Hardy [4] is the sharpest result (as of 1916) for no derivative, *finite or infinite*. (Hardy showed $b > 1$, $ab \geqq 1$ sufficient to establish the non-existence of any *finite* derivative. He also showed that those same conditions, together with $a(b + 1) < 2$ for $b = 4k + 1$, permitted the existence of an *infinite* derivative at certain points.) To illustrate the difference between (2) and (3) for $a = \frac{1}{2}$, (2) requires $b \geqq 13$, while (3) permits $b = 7$. However, as far as the authors know there may be considerable work to be done in the direction of lowering the bound of $1 + \frac{3\pi}{2}(1 - a)$ in (3) for the case of no derivative, finite or infinite.

Owing to the unusual nature of $W(a, b, x)$ and the absence of any previous table, or even graph, despite the countless number of theoretical papers, it was believed that an extensive table of this Weierstrass function for some typical pair of parameters a and b might be of value as more than a mere curiosity, namely for suggesting or motivating further research, and for its interest to workers in numerical analysis. Thus, in this last connection, it might be of interest to determine empirically what results in numerical integration and possibly interpolation are available from the continuity alone. That $W(a, b, x)$ is integrable follows from its continuity, and one might be curious to see the results of applying standard numerical integration formulas where the usual derivative formulas for the remainder would be inapplicable. Likewise, one might be curious to test out standard Lagrangian interpolation, where the remainder is often expressed in terms of derivatives. (We can write down interpolation and numerical integration formulas, avoiding derivatives in the remainder terms by employing divided differences and integrals with divided differences in the integrand, respectively. However, one usually estimates divided differences in terms of derivatives.) Finally, one's curiosity might extend as far as

Received February 23, 1960; revised July 28, 1960.

glancing at the results of standard numerical differentiation and interpretation of the results in the light of the knowledge that $W(a, b, x)$ has no derivative.

For tabulation of any $W(a, b, x)$, it is immediately apparent from (1) that

$$W(a, b, 1 + x) = -W(a, b, x), \tag{4}$$

so that the range of x need not go outside (0, 1). From (1),

$$\begin{aligned} W(a, b, 0) &= -W(a, b, 1) = a/(1 - a); \\ W(a, b, \tfrac{1}{2}) &= 0. \end{aligned} \tag{5}$$

From the trigonometric identity

$$\cos(m\pi(\tfrac{1}{2} \pm t)) = \mp(-1)^{(m-1)/2} \sin m\pi t, \ m \text{ odd}, \tag{6}$$

we have

$$W(a, b, \tfrac{1}{2} + t) = -W(a, b, \tfrac{1}{2} - t), \tag{7}$$

so that for complete tabulation of any $W(a, b, x)$ it suffices for x to range from 0 to $\frac{1}{2}$.

In connection with the choice of a and b, it is apparent that for a close to 1, we can choose b as low as 3, but the convergence of the series in (1) would be too slow for practical calculation of $W(a, b, x)$ to high accuracy. Making a very small would give rapid convergence, but for accuracy fixed at a certain number of decimal places as a tends to get very small, say

$$a = \epsilon, \qquad b^n > N = \left\{1 + \frac{3\pi}{2}(1 - \epsilon)\right\}^n \Big/ \epsilon^n$$

becomes enormous and $W(\epsilon, b, x)$ becomes essentially the first term of (1), $\epsilon \cos (b^n \pi x)$, whose graph would appear like that of a very highly oscillatory function of small amplitude. As a compromise between these two extreme types, we took $a = \frac{1}{2}$ and $b = 7$. The choice $a = \frac{1}{2}$ did not lead to too many terms of (1), 50 terms giving a truncating error $< \frac{1}{2} \cdot 10^{-15}$, and yet there were sufficient terms beyond the first few to give a graph that is characteristic of $W(a, b, x)$ rather than a predominantly sinusoidal type of curve. The $b = 7$ barely satisfies (3), thus tending to minimize the oscillatory behavior of $W(a, b, x)$ and to facilitate graphing. We shall denote $W(a, b, x)$ which is tabulated here for $a = \frac{1}{2}$ and $b = 7$ by $W(x)$.

This present table of $W(x)$, $x = 0(.001)1$ to 12D, was printed out and rounded from a preliminary calculation on the IBM 704 to several more places. Two separate and independent print-outs, supposedly identical, were proofread against each other, with just a single print-out error turning up. Naturally, no differencing check could be made upon the correctness of this table of $W(x)$, but every value underwent the following final functional check:

$$W(7x) = 2W(x) - \cos(7\pi x), \tag{8}$$

which was performed by desk calculation upon $W(x)$ on one of the preliminary print-outs. The results showed $W(x)$ to be correct to around 14D. In employing (8), $W(7x)$ was found in the table as $\pm W(x')$ for some suitable x', $0 \leq x' \leq \frac{1}{2}$, according to (4) and (7), and $\cos(7\pi x)$, after reduction of $7\pi x$ to the first quadrant, was

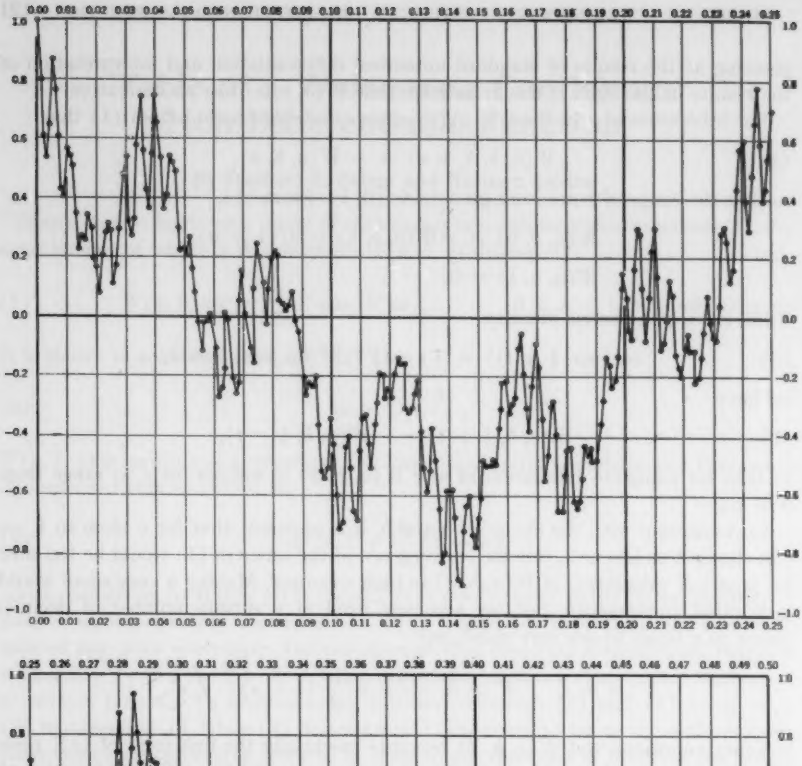

Fig. 1.—Illustration of a Weierstrass, Everywhere-Continuous Nowhere-Differentiable Function, $W(x) = \sum_{n=1}^{\infty} a^n \cos(b^n \pi x)$ $a = \frac{1}{2}$; $b = 7$; $x = 0(0.001)0.500$

looked up in a well-known 15-place table at intervals of 0.01° [5]. The final 12-decimal table was checked by reading it several times against one of the print-outs, and it is believed to be correct to well within a unit in the 12th decimal.

The purpose of the accompanying figure, which is merely a broken line graph of the table of $W(x)$, is to furnish at a glance a view of the peculiar behavior of $W(x)$. Of course, the graphical picture would be more complete if the time and means were available for calculating $W(a, b, x)$ as a function of a also, and for a sequence of permissible odd integral values of b (according to (3)) to correspond to each a. Although no offhand justification could be found for drawing anything smoother than a broken line connecting these 500 points, one still finds its ripples of irregularity, superposed upon a broader pattern of smoothness, to be quite revealing as to the nature of $W(x)$ and how it might appear under repeated "magnification" (i.e., subtabulation).

To establish (8), replace x by $7x$, in $W(x) = \sum_{n=1}^{\infty} \cos(7^n\pi x)/2^n$, to get

$$W(7x) = 2\sum_{n=1}^{\infty} \cos(7^{n+1}\pi x)/2^{n+1} = 2\sum_{n'=2}^{\infty} \cos(7^{n'}\pi x)/2^{n'} = 2W(x) - \cos(7\pi x).$$

By repeated application of (8),

$$\begin{aligned} W(7^n x) &= 2W(7^{n-1}x) - \cos(7^n\pi x) = 4W(7^{n-2}x) - 2\cos(7^{n-1}\pi x) - \cos(7^n\pi x) \\ &= 8W(7^{n-3}x) - 4\cos(7^{n-2}\pi x) - \cdots \quad \text{etc. until we reach} \end{aligned}$$

$$W(7^n x) = 2^n W(x) - \sum_{r=0}^{n-1} 2^r \cos(7^{n-r}\pi x). \tag{9}$$

From (9), for $x = 1/7^n$, $W(1) = -1 = 2^n W(1/7^n) - \sum_{r=0}^{n-1} 2^r \cos(\pi/7^r)$, from which

$$W(1/7^n) = -1/2^{n-1} + \sum_{r=1}^{n-1} \cos(\pi/7^r)/2^{n-r}. \tag{10}$$

Letting $n \longrightarrow \infty$ in (10), we see at once that

$$\lim_{n\to\infty} \left\{ \sum_{r=1}^{n-1} 2^r \cos(\pi/7^r) \right\} \Big/ 2^n = 1. \tag{11}$$

To test the value of standard numerical integration formulas upon $W(x)$, whose integral is given by

$$\int_0^x W(t)\,dt = \frac{1}{\pi}\sum_{n=1}^{\infty} \sin(7^n\pi x)/14^n, \tag{12}$$

the values of $\int_0^{0.1} W(t)dt, \int_{0.1}^{0.2} W(t)dt, \cdots, \int_{0.4}^{0.5} W(t)dt$ were computed analytically from (12), and then were computed numerically by both trapezoidal and Simpson's rules at intervals of 0.001, with the following results:

Interval	True Value	Trapezoidal Rule	Deviation	Simpson's Rule	Deviation
0 to 0.1	0.01899 29	0.01898 76	−0.00000 53	0.01901 44	+0.00002 15
0.1 to 0.2	−0.04145 65	−0.04143 80	+0.00001 85	−0.04145 43	+0.00000 22
0.2 to 0.3	0.03084 62	0.03084 43	−0.00000 19	0.03085 14	+0.00000 52
0.3 to 0.4	0.00337 70	0.00342 54	+0.00004 84	0.00340 27	+0.00002 57
0.4 to 0.5	−0.03298 02	−0.03300 67	−0.00002 65	−0.03288 27	+0.00009 75

The results show no recognizable advantage in Simpson's rule. In fact, the sum of the absolute values of the above deviations in the trapezoidal rule is around 10^{-4}, while the sum of the absolute values of the Simpson deviations is around $1\frac{1}{2}\cdot 10^{-4}$. This may indicate that no higher-point formula will improve over the trapezoidal formula.

Lagrangian polynomial interpolation at intervals of 0.002 was tried for the 2- through 7-point cases, for a mid-interval (i.e., already tabulated) value of $W(x)$ at two different places, $x = 0.007$ and $x = 0.037$, where the true value to 5D is 0.60807 and 0.43362 respectively. At each place the error in almost all cases ranged from around 0.01 to 0.05. More specifically, for $x = 0.007$ the error fluctuated between 0.01 for every even-point interpolation and 0.014 to 0.049 for various odd-point interpolations, and for $x = 0.037$ there were deviations of 0.032 and 0.055 for respective 2-point and 3-point interpolation and deviations ranging from 0.001 to 0.021 in the higher-point interpolation. On the basis of these two tests alone it would appear that one could not really count upon any systematic improvement beyond linear interpolation.

Finally, out of pure curiosity, 2- through 7-point Lagrangian differentiation, for the "first derivative," was tried out at the tabular interval of 0.001, for $x = 0.002$, and surprisingly enough, outside of the 2-point answer of -74 and the 3-point answer of -133, the remaining four cases all came within 6 units of -150.

From a casual look at the graph of $W(x)$, it is apparent that in place of the derivative there is a general directional trend from any point x_0 if we do not go too far away from x_0, and we might seek a suitable quantitative estimate for an "average slope" between x_0 and $x_0 + h$. (The discussion here is concerned with a suitable generalization of the left- or right-hand derivative, rather than the derivative.) One suggestion that would appear natural for $W(x, a, b)$, or any other continuous function, would be to investigate the possibilities of the average of the difference quotient $\{f(x) - f(x_0)\}/(x - x_0)$, which exists and is itself continuous for every x except x_0 in the open interval $(x_0, x_0 + h)$. This average difference quotient or $\mathfrak{D}_h f(x_0)$ might have the following definition (assuming that it exists in the first place):

$$\mathfrak{D}_h f(x_0) = \frac{1}{h}\int_{x_0}^{x_0+h} \{[f(x) - f(x_0)]/(x - x_0)\}\, dx. \tag{13}$$

That (13) may be a suitable generalization follows from the fact that when $f'(x_0)$ exists, (13) exists, and

$$\lim_{h\to 0} \mathfrak{D}_h f(x_0) = f'(x_0). \tag{14}$$

This is seen at once from the replacement of $\{f(x) - f(x_0)\}/(x - x_0)$ by $f'(x_0) + \epsilon(x)$ in (13) and the continuity of $\epsilon(x)$ in the *closed* set $(x_0, x_0 + h)$ which makes $\epsilon(x)$ integrable. Thus (13) exists and

$$\left|\frac{1}{h}\int_{x_0}^{x_0+h} \epsilon(x)\, dx\right| \to 0 \quad \text{as} \quad h \to 0,$$

which implies (14).

It is not difficult to find examples of continuous functions $f(x)$ where $f'(x_0)$ does not exist and (a) also $\mathfrak{D}_h f(x_0)$ does not exist, or (b) $\mathfrak{D}_h f(x_0)$ exists but $\lim_{h\to 0} \mathfrak{D}_h f(x_0)$ does not exist. But we may also have (c) no $f'(x_0)$, with both $\mathfrak{D}_h f(x_0)$ and $\lim_{h\to 0} \mathfrak{D}_h f(x_0)$ existing. In other words the existence of $\lim_{h\to 0} \mathfrak{D}_h f(x_0)$ still

does not imply the existence of $f'(x_0)$. Such a counter-example,* which is due to the referee, is the following. Let $x_0 = 0$ and

$$f(x) = x \sin \frac{1}{x} \qquad (x \neq 0)$$
$$f(0) = 0.$$

This continuous function has no derivative at $x = 0$, but

$$\lim_{h \to 0} \mathfrak{D}_h f(0) = 0.$$

First

$$\mathfrak{D}_h = \frac{1}{h} \int_0^h \sin\left(\frac{1}{x}\right) dx$$

exists since the integrand is bounded and continuous except at one point. This suffices. To estimate $\mathfrak{D}_h$ we let

$$I_n = \int_{1/(n+1)\pi}^{1/n\pi} \sin\left(\frac{1}{x}\right) dx = \int_{n\pi}^{(n+1)\pi} \frac{1}{y^2} \sin y \, dy.$$

By the mean value theorem

$$I_n = (-1)^n \cdot 2/\theta_n^2$$

where

$$n\pi < \theta_n < (n+1)\pi.$$

Suppose that $h = 1/(n+a)\pi, 0 \leqq a < 1$. Then

$$\mathfrak{D}_h = (n+a)\pi \left[\int_{(n+a)\pi}^{(n+1)\pi} y^{-2} \sin y \, dy + I_{n+1} + I_{n+2} + \cdots \right],$$

and therefore $| \mathfrak{D}_h | < (n+a)\pi \, | I_n | < 2(n+a)\pi/n^2\pi^2$.

Therefore as $h \to 0$, $\mathfrak{D}_h$ also $\to 0$.

The authors wish to acknowledge the assistance of Mrs. Charlene M. Janos in checking the entire table of $W(x)$ by the functional check (8).

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* Another counter-example found after that of the referee is the following: $f(x) = x\phi(x)$, $x \neq 0$, $f(0) = 0$, where $\phi(x) = 1$ except in the intervals $[(1/n - 1/n^3), 1/n]$, within which $\phi(x) = 0$. Now $f(x)$ is continuous at $x = 0$ and has no derivative there. But $1/h \int_0^h \phi(x) dx \to 1$ as $h \to 0$, because the "dipped-out" area becomes an infinitesimal fraction of the whole (also infinitesimal) area between 0 and h, since as $h \sim 1/n$, we remove $\sum_{m=n}^{\infty} 1/m^3 \sim 1/2n^2 \sim 0(h)$.

Table of $W(x) \equiv \sum_{n=1}^{\infty} \cos(7^n \pi x)/2^n$

x	$W(x)$		x	$W(x)$	
.000	1.00000 00000 00	1.000	.050	.23088 91433 53	.950
.001	.80391 58298 49	.999	.051	.20682 52628 39	.949
.002	.61188 60438 58	.998	.052	.27128 71570 31	.948
.003	.53777 60375 27	.997	.053	.16118 34941 71	.947
.004	.64747 48039 38	.996	.054	.08069 56769 70	.946
.005	.87163 69853 23	.995	.055	−.02066 12990 04	.945
.006	.76687 71957 75	.994	.056	−.11450 55193 73	.944
.007	.60807 34552 61	.993	.057	−.02295 65257 19	.943
.008	.43502 94075 78	.992	.058	−.01951 72464 55	.942
.009	.40541 06494 76	.991	.059	.01151 68818 80	.941
.010	.56641 31472 93	.990	.060	−.09698 14952 80	.940
.011	.54275 36720 27	.989	.061	−.27187 35472 72	.939
.012	.50694 91215 98	.988	.062	−.23653 00063 45	.938
.013	.34801 25245 87	.987	.063	−.16965 63244 21	.937
.014	.22473 91530 39	.986	.064	.01498 43634 87	.936
.015	.27196 45026 68	.985	.065	−.00239 70885 80	.935
.016	.25665 87904 18	.984	.066	−.20181 95236 74	.934
.017	.34500 48434 78	.983	.067	−.25856 31395 23	.933
.018	.29744 09740 20	.982	.068	−.21932 57817 04	.932
.019	.19896 02842 26	.981	.069	.05550 33815 80	.931
.020	.16232 54753 01	.980	.070	.15690 95326 47	.930
.021	.07772 75335 97	.979	.071	.01436 83992 20	.929
.022	.20584 44795 34	.978	.072	−.09812 03304 88	.928
.023	.28363 56796 36	.977	.073	−.15074 56668 50	.927
.024	.31741 47365 60	.976	.074	.09240 88499 35	.926
.025	.28730 16038 97	.975	.075	.24890 58340 07	.925
.026	.11054 29341 42	.974	.076	.20632 59257 00	.924
.027	.17279 94307 92	.973	.077	.11462 33882 35	.923
.028	.29881 59987 33	.972	.078	−.02304 18919 60	.922
.029	.48372 84610 58	.971	.079	.09557 81653 49	.921
.030	.54441 21945 09	.970	.080	.19794 01773 19	.920
.031	.32388 99122 78	.969	.081	.22531 98834 20	.919
.032	.26990 13283 84	.968	.082	.20876 59176 94	.918
.033	.33225 74462 04	.967	.083	.05397 57757 43	.917
.034	.58370 92580 29	.966	.084	.04851 66043 63	.916
.035	.74931 30151 91	.965	.085	.02128 19742 78	.915
.036	.56632 09611 85	.964	.086	.03684 58507 18	.914
.037	.43361 86486 58	.963	.087	.08433 37682 49	.913
.038	.36496 26383 50	.962	.088	−.01214 33731 42	.912
.039	.55435 98106 40	.961	.089	−.05215 21171 01	.911
.040	.75565 42269 37	.960	.090	−.19576 22844 41	.910
.041	.66141 61182 70	.959	.091	−.26338 74226 81	.909
.042	.54244 67604 80	.958	.092	−.21893 60178 36	.908
.043	.36553 53065 06	.957	.093	−.22931 50089 58	.907
.044	.41175 90597 74	.956	.094	−.19289 54543 34	.906
.045	.54502 46829 96	.955	.095	−.36048 94459 76	.905
.046	.52178 60105 03	.954	.096	−.51624 68304 42	.904
.047	.49248 88676 02	.953	.097	−.54350 09214 87	.903
.048	.30088 64437 50	.952	.098	−.50350 10354 40	.902
.049	.22797 19914 12	.951	.099	−.33848 06786 76	.901
	$-W(x)$	x		$-W(x)$	x

TABLE OF $W(x)$—*Continued*

x	$W(x)$		x	$W(x)$	
.100	−.42532 54041 76	.900	.150	−.60928 87419 74	.850
.101	−.60122 07728 99	.899	.151	−.48052 65499 66	.849
.102	−.71436 04664 23	.898	.152	−.49741 97079 36	.848
.103	−.69032 26595 69	.897	.153	−.49479 75145 65	.847
.104	−.43794 73064 53	.896	.154	−.47354 23522 65	.846
.105	−.40215 63534 91	.895	.155	−.49291 96963 68	.845
.106	−.50671 30937 48	.894	.156	−.36979 28855 91	.844
.107	−.65237 17461 67	.893	.157	−.30677 09741 35	.843
.108	−.68741 46475 51	.892	.158	−.21917 49907 16	.842
.109	−.44815 12393 09	.891	.159	−.21531 69983 41	.841
.110	−.33948 90492 28	.890	.160	−.32774 87639 11	.840
.111	−.32696 83696 91	.889	.161	−.30751 75250 62	.839
.112	−.42541 62768 18	.888	.162	−.27020 56659 27	.838
.113	−.50701 25026 15	.887	.163	−.11494 78174 84	.837
.114	−.36527 04053 46	.886	.164	−.05693 76441 49	.836
.115	−.28717 02983 14	.885	.165	−.19542 29668 60	.835
.116	−.19435 86539 72	.884	.166	−.30152 83944 41	.834
.117	−.20343 01549 89	.883	.167	−.38366 39179 31	.833
.118	−.27648 95287 96	.882	.168	−.22909 39440 73	.832
.119	−.24091 87061 61	.881	.169	−.09500 91416 95	.831
.120	−.27427 89580 66	.880	.170	−.17303 80098 99	.830
.121	−.19594 53705 68	.879	.171	−.33806 67973 86	.829
.122	−.14745 62719 19	.878	.172	−.55064 02832 16	.828
.123	−.16077 90760 02	.877	.173	−.46584 78110 99	.827
.124	−.16313 11716 57	.876	.174	−.30129 83741 07	.826
.125	−.30795 98441 70	.875	.175	−.27803 69565 60	.825
.126	−.32779 07918 31	.874	.176	−.39280 65936 50	.824
.127	−.30642 17906 98	.873	.177	−.65182 79852 26	.823
.128	−.25457 54992 71	.872	.178	−.65182 39388 20	.822
.129	−.20521 28645 40	.871	.179	−.53351 02371 89	.821
.130	−.38226 57006 18	.870	.180	−.44268 44274 72	.820
.131	−.51008 60535 12	.869	.181	−.43578 01021 83	.819
.132	−.58897 88293 07	.868	.182	−.61985 87140 15	.818
.133	−.51605 03218 22	.867	.183	−.64922 21692 63	.817
.134	−.37228 02555 49	.866	.184	−.62277 87433 01	.816
.135	−.48222 40135 76	.865	.185	−.54441 65431 73	.815
.136	−.64476 90942 44	.864	.186	−.43416 51101 85	.814
.137	−.82656 60891 90	.863	.187	−.47153 43722 97	.813
.138	−.78900 50242 13	.862	.188	−.44187 85912 52	.812
.139	−.58460 20582 18	.861	.189	−.48324 34653 21	.811
.140	−.58017 61018 65	.860	.190	−.48100 45544 76	.810
.141	−.67358 93294 65	.859	.191	−.36210 24639 76	.809
.142	−.88334 97740 78	.858	.192	−.28385 07250 58	.808
.143	−.90195 54475 25	.857	.193	−.13957 34378 96	.807
.144	−.71735 67984 31	.856	.194	−.17132 73511 09	.806
.145	−.63542 71888 15	.855	.195	−.24380 91207 97	.805
.146	−.60172 84929 47	.854	.196	−.21870 72532 52	.804
.147	−.73979 05811 32	.853	.197	−.13838 71322 99	.803
.148	−.77996 61358 53	.852	.198	.09801 82360 71	.802
.149	−.67821 23623 64	.851	.199	.14666 52327 61	.801
	$-W(x)$	x		$-W(x)$	x

TABLE OF $W(x)$—*Continued*

x	$W(x)$		x	$W(x)$	
.200	.06366 10018 75	.800	.250	.70710 67811 87	.750
.201	−.04175 22364 04	.799	.251	.59986 16383 91	.749
.202	−.07476 70750 18	.798	.252	.42999 19525 71	.748
.203	.16401 55220 04	.797	.253	.36660 93235 94	.747
.204	.29971 35815 61	.796	.254	.32795 62544 06	.746
.205	.28236 84375 00	.795	.255	.45218 11134 91	.745
.206	.10035 12674 93	.794	.256	.43000 78607 24	.744
.207	−.08006 36504 29	.793	.257	.38270 70934 21	.743
.208	.06602 53228 00	.792	.258	.32431 23791 08	.742
.209	.22194 83909 79	.791	.259	.18040 34507 72	.741
.210	.29866 93766 43	.790	.260	.20082 17490 15	.740
.211	.14347 00102 07	.789	.261	.19502 92282 86	.739
.212	−.11156 27605 97	.788	.262	.28277 11533 62	.738
.213	−.08927 61047 33	.787	.263	.33124 94143 10	.737
.214	−.00503 43683 93	.786	.264	.19535 83704 39	.736
.215	.12393 18196 44	.785	.265	.13130 01934 73	.735
.216	.07237 02520 68	.784	.266	.05923 97001 64	.734
.217	−.12870 66881 32	.783	.267	.20320 81841 97	.733
.218	−.17190 63119 85	.782	.268	.38097 30175 89	.732
.219	−.20539 46224 39	.781	.269	.36488 72388 22	.731
.220	−.10396 93551 29	.780	.270	.30069 58598 63	.730
.221	−.05670 84900 53	.779	.271	.12888 96220 58	.729
.222	−.11058 04979 98	.778	.272	.21930 88571 52	.728
.223	−.11053 47920 41	.777	.273	.45450 66523 44	.727
.224	−.22263 98542 44	.776	.274	.58788 37027 28	.726
.225	−.20433 20524 80	.775	.275	.61063 78772 02	.725
.226	−.13899 14773 88	.774	.276	.37897 11709 32	.724
.227	−.05338 52274 30	.773	.277	.35491 00906 37	.723
.228	.07179 91975 44	.772	.278	.53317 01360 59	.722
.229	−.01896 62898 75	.771	.279	.74080 38401 87	.721
.230	−.07113 29711 74	.770	.280	.87388 44641 26	.720
.231	−.08351 49064 96	.769	.281	.66344 41290 97	.719
.232	.03967 20418 75	.768	.282	.55360 44310 88	.718
.233	.28470 23445 34	.767	.283	.59858 96747 52	.717
.234	.30373 06421 48	.766	.284	.75311 92971 19	.716
.235	.25331 56946 85	.765	.285	.93575 68089 02	.715
.236	.12247 07877 84	.764	.286	.80593 31523 57	.714
.237	.16091 46178 36	.763	.287	.70250 54785 63	.713
.238	.43416 31698 35	.762	.288	.62769 78735 12	.712
.239	.57254 03757 22	.761	.289	.64051 31524 34	.711
.240	.60020 74057 93	.760	.290	.76998 70795 56	.710
.241	.39293 37880 48	.759	.291	.71343 52538 50	.709
.242	.29091 21091 20	.758	.292	.70111 02427 36	.708
.243	.47723 86339 28	.757	.293	.59700 86384 17	.707
.244	.65452 02702 15	.756	.294	.48196 39792 70	.706
.245	.78049 97326 19	.755	.295	.48841 28610 21	.705
.246	.58771 13946 18	.754	.296	.43820 32711 49	.704
.247	.39392 08421 96	.753	.297	.53246 89138 76	.703
.248	.43534 55892 80	.752	.298	.50028 47645 04	.702
.249	.53704 70311 88	.751	.299	.36415 33185 00	.701
	$-W(x)$	x		$-W(x)$	x

TABLE OF $W(x)$—*Continued*

x	$W(x)$		x	$W(x)$	
.300	.26286 55560 60	.700	.350	.00778 77869 67	.650
.301	.14582 98589 87	.699	.351	−.17204 22086 50	.649
.302	.28563 97407 01	.698	.352	−.23029 21241 86	.648
.303	.36633 50157 87	.697	.353	−.23647 30864 44	.647
.304	.33282 81740 47	.696	.354	−.01762 25274 42	.646
.305	.21458 96315 30	.695	.355	.06669 22195 13	.645
.306	.00939 20926 45	.694	.356	.00075 62690 21	.644
.307	.10710 80078 80	.693	.357	−.02063 75236 12	.643
.308	.25624 91704 22	.692	.358	−.08880 13434 64	.642
.309	.37838 97945 27	.691	.359	.03980 77688 40	.641
.310	.34385 20085 52	.690	.360	.10006 60841 69	.640
.311	.09875 62075 19	.689	.361	.12219 89553 34	.639
.312	.10737 46483 70	.688	.362	.18737 55558 83	.638
.313	.23160 40973 59	.687	.363	.10954 21317 00	.637
.314	.45535 62002 52	.686	.364	.12819 87330 70	.636
.315	.54102 65479 93	.685	.365	.07610 38829 89	.635
.316	.33736 28357 70	.684	.366	.09741 56111 37	.634
.317	.28355 86706 20	.683	.367	.22835 66376 41	.633
.318	.30968 52189 51	.682	.368	.19796 35878 29	.632
.319	.51427 37371 70	.681	.369	.16937 55518 30	.631
.320	.66458 80166 07	.680	.370	−.00562 33217 53	.630
.321	.55383 02222 36	.679	.371	−.07189 57360 01	.629
.322	.51306 08367 00	.678	.372	.04604 08396 66	.628
.323	.43864 14186 98	.677	.373	.07904 58408 70	.627
.324	.52349 24425 12	.676	.374	.09708 84764 28	.626
.325	.63004 29627 66	.675	.375	−.12756 11441 22	.625
.326	.59308 72965 01	.674	.376	−.30043 50117 72	.624
.327	.62794 97613 07	.673	.377	−.27256 67762 10	.623
.328	.51805 41271 03	.672	.378	−.21256 78517 98	.622
.329	.47323 47350 89	.671	.379	−.09426 94456 39	.621
.330	.45296 76932 02	.670	.380	−.26632 63801 78	.620
.331	.41365 95766 99	.669	.381	−.48045 19086 93	.619
.332	.52433 35362 36	.668	.382	−.55645 23195 61	.618
.333	.45999 71920 26	.667	.383	−.52637 52089 99	.617
.334	.37014 12343 31	.666	.384	−.33212 51324 29	.616
.335	.22552 02235 19	.665	.385	−.39479 74417 87	.615
.336	.10904 55459 52	.664	.386	−.54657 07923 33	.614
.337	.23421 37763 08	.663	.387	−.66192 84709 16	.613
.338	.25303 23429 37	.662	.388	−.69040 53292 05	.612
.339	.23376 32937 51	.661	.389	−.49978 37896 32	.611
.340	.05089 90862 54	.660	.390	−.48100 38625 93	.610
.341	−.15716 90669 72	.659	.391	−.50444 79258 90	.609
.342	−.09044 72949 99	.658	.392	−.56183 05832 68	.608
.343	−.01752 94215 89	.657	.393	−.62318 58803 80	.607
.344	.09699 75179 51	.656	.394	−.50778 26196 53	.606
.345	−.01688 93672 23	.655	.395	−.49647 33948 32	.605
.346	−.25654 09168 79	.654	.396	−.41302 52891 40	.604
.347	−.27382 88597 64	.653	.397	−.35589 92135 83	.603
.348	−.21463 82850 53	.652	.398	−.38867 82210 92	.602
.349	.00184 25835 82	.651	.399	−.35791 63481 26	.601
	$-W(x)$	x		$-W(x)$	x

TABLE OF $W(x)$—*Concluded*

x	$W(x)$		x	$W(x)$	
.400	−.43633 89981 25	.600	.450	−.14085 88911 07	.550
.401	−.34108 64853 67	.599	.451	−.28701 85703 94	.549
.402	−.19995 66617 14	.598	.452	−.40662 30552 60	.548
.403	−.15624 84342 39	.597	.453	−.26053 58919 44	.547
.404	−.15344 31864 89	.596	.454	−.09792 69986 33	.546
.405	−.33660 42737 89	.595	.455	−.14569 54090 97	.545
.406	−.33007 06597 37	.594	.456	−.26984 09362 85	.544
.407	−.20452 46632 06	.593	.457	−.47876 24067 97	.543
.408	−.09102 42265 68	.592	.458	−.44907 28535 16	.542
.409	−.04111 67965 83	.591	.459	−.34753 36936 00	.541
.410	−.26774 44625 33	.590	.460	−.33327 19438 23	.540
.411	−.38274 33375 09	.589	.461	−.35146 25764 11	.539
.412	−.36998 51442 42	.588	.462	−.52273 62074 43	.538
.413	−.24689 45307 00	.587	.463	−.55987 41267 67	.537
.414	−.11736 78392 84	.586	.464	−.57344 06332 45	.536
.415	−.29765 17844 11	.585	.465	−.57404 69550 42	.535
.416	−.47494 21492 22	.584	.466	−.48136 22660 42	.534
.417	−.59655 84918 74	.583	.467	−.51427 61901 27	.533
.418	−.53275 36804 59	.582	.468	−.51301 35662 51	.532
.419	−.35968 31001 54	.581	.469	−.61436 17729 69	.531
.420	−.44265 28777 24	.580	.470	−.69144 70666 05	.530
.421	−.57367 68152 00	.579	.471	−.56392 63767 92	.529
.422	−.75568 10645 53	.578	.472	−.45520 04387 35	.528
.423	−.77343 16806 93	.577	.473	−.32911 92328 10	.527
.424	−.63242 61726 82	.576	.474	−.42540 32355 39	.526
.425	−.64210 94688 15	.575	.475	−.57627 07637 41	.525
.426	−.64792 12620 64	.574	.476	−.51397 83689 36	.524
.427	−.77107 83101 91	.573	.477	−.35913 37922 98	.523
.428	−.82369 79240 71	.572	.478	−.10430 50805 99	.522
.429	−.76886 58123 18	.571	.479	−.10454 99179 66	.521
.430	−.78295 98538 29	.570	.480	−.26292 26878 67	.520
.431	−.67178 95122 99	.569	.481	−.31706 81976 54	.519
.432	−.65728 68184 10	.568	.482	−.24133 62343 53	.518
.433	−.65957 56572 42	.567	.483	.05762 50734 01	.517
.434	−.67892 26214 60	.566	.484	.17288 12030 15	.516
.435	−.76752 41703 56	.565	.485	.08627 79883 17	.515
.436	−.62898 65506 20	.564	.486	−.05153 93039 59	.514
.437	−.49719 39800 06	.563	.487	−.12044 48896 59	.513
.438	−.38838 06903 26	.562	.488	.10705 03214 66	.512
.439	−.41153 59153 14	.561	.489	.26694 06923 26	.511
.440	−.58326 16692 24	.560	.490	.28240 83062 16	.510
.441	−.52360 70974 04	.559	.491	.15028 16426 95	.509
.442	−.38190 82986 12	.558	.492	−.02361 75574 02	.508
.443	−.17443 01231 01	.557	.493	.06684 38933 29	.507
.444	−.12778 79441 71	.556	.494	.15875 42472 12	.506
.445	−.32413 68662 58	.555	.495	.23215 63219 75	.505
.446	−.38999 44407 08	.554	.496	.18367 46210 92	.504
.447	−.35689 24535 27	.553	.497	.01931 21600 07	.503
.448	−.13266 84382 20	.552	.498	.00378 55928 21	.502
.449	.00059 06984 60	.551	.499	−.04441 66347 11	.501
			.500	.00000 00000 00	.500
	$-W(x)$	x		$-W(x)$	x

On the Numerical Solution of Convolution Integral Equations and Systems of such Equations

By J. G. Jones

1. Introduction. This paper discusses the application of a simple quadrature formula to the numerical solution of convolution integral equations of Volterra type and to systems of simultaneous equations of the same type. The convergence of the processes is considered in some detail, proofs being given that at a fixed value of the independent variable the errors in the solution tend to zero as the step length tends to zero.

2. Types of Equations Considered. The convolution integral equation to be solved is

$$ag(x) - \int_0^x W(x - \xi)g(\xi)\, d\xi = f(x) \tag{1}$$

where 'a' is a constant, and f, W are given functions. The cases $a = 0$ (equation of the first kind) and $a \neq 0$ (equation of the second kind) are discussed.

The corresponding system of equations can be written

$$\mathbf{A}\mathbf{g}(x) - \int_0^x \mathbf{W}(x - \xi)\mathbf{g}(\xi)\, d\xi = \mathbf{f}(x) \tag{2}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \text{a matrix of constants,}$$

$$\mathbf{W}(z) = \begin{bmatrix} W_{11}(z) \cdots & W_{1n}(z) \\ \vdots & \vdots \\ W_{n1}(z) \cdots & W_{nn}(z) \end{bmatrix}, \mathbf{f}(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}, \mathbf{g}(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{bmatrix}$$

and the notation involving integration is interpreted in an obvious way. For this system of equations the rank of **A** is an important parameter.

Equation (1) is well known, commonly occurring in practical problems. The system of equations (2) does not appear to have been previously discussed, so an instance of a practical problem in which it arises is briefly described.

In the linearized supersonic theory of [1] the system of equations (2) gives the pressure coefficients $f_i(x)$ on n spanwise wing stations produced by a quasi-cylindrical shaped fuselage defined by n Fourier components $g_i(x)$. The $W_{ij}(z)$ are tabulated influence functions and **A** is of rank unity. In [2] the problem of determining the fuselage shape required to produce prescribed pressure coefficients on wing stations is considered. Special attention is there devoted to the numerical solution of (2) when n equals 2 and the rank of **A** is unity.

Received March 31, 1959; revised November 16, 1959.

3. The Use of the Trapezoidal Rule. In the method to be described, the equations are solved by dividing the x-axis into equal small intervals, replacing the integrals in the equations by the corresponding approximate expressions given by the trapezoidal rule, and hence deriving a formula for the step-by-step solution.

The solution of an integral equation by replacing the integral involved by a quadrature formula may be regarded as a standard method; see, for example, [3]. However, adequate discussions of the convergence of such numerical methods appear to be lacking. This paper is concerned with the convergence of the method in the particular case of the application of the trapezoidal rule to equation (1), and with the extension of the method to the solution of the system of equations (2).

The trapezoidal rule is the least accurate of quadrature formulas for a given step size, but it has the computational advantage of leading to a simple routine suitable for either desk or automatic computing, and furthermore is the most simple case to consider as regards convergence.

Subject to assumed bounds on the derivatives of the functions involved, it has been found possible to establish a bound for the truncation error at a given value of the independent variable, and hence to prove the convergence of the process.

The method of establishing a bound is not intended as a practical means of estimating errors. Practical ways of checking the accuracy of a numerical computation of this type are either to repeat the solution using a different step length, or to evaluate the integrals in the equations by using the numerical solution and a more accurate quadrature formula than the trapezoidal rule. However, it is shown in Section 8 that for equations of the first kind the numerical method described may in some cases also provide a convenient means of error analysis.

4. Sources of Error in the Solution. Consider the truncation error in this type of numerical solution. In going from the mth step to the $(m + 1)$th, errors occur from two sources (neglecting rounding errors which it is assumed throughout are kept smaller than the truncation errors):

(i) due to replacing the integrals by a quadrature formula

(ii) due to the fact that the values of the solution at the first m steps, all of which go into the quadrature formula for the evaluation of the solution at the $(m + 1)$th step, have corresponding errors in them.

In the following treatment the error at a fixed value of x is considered and the step length is taken as an integral fraction of x. As the step length decreases, the decrease in the corresponding error produced in a single step is offset by the fact that the number of steps required to reach x increases inversely. An analogous error analysis for the numerical solution of an ordinary differential equation is given in [4]. The form of the bound obtained in this latter case is similar to that of the bound described in Section 6.

5. The Single Integral Equation. In this section we consider the numerical solution of (1). If $a \neq 0$ the step-by-step process is started by means of the equation

$$ag(0) = f(0). \tag{3}$$

If $a = 0$, (1) gives, on differentiation,

$$-W(0)g(x) - \int_0^x W'(x-\xi)g(\xi)\,d\xi = f'(x). \tag{4}$$

If now $W(0) \neq 0$, the step-by-step process is started by means of the equation

$$-W(0)g(0) = f'(0). \tag{5}$$

If $W(0) = 0$, further differentiation $(n + 1)$ times is required until we reach a derivative $W^{(n)}(0) \neq 0$.

Equal intervals of length δ are now taken along the x-axis. For the mth step we have

$$ag(m\delta) - \int_0^{m\delta} W(m\delta - \xi)g(\xi)\,d\xi = f(m\delta).$$

Replacing the integral by the corresponding trapezoidal rule approximation and rearranging:

$$\begin{aligned}\left\{a - \frac{\delta}{2}W(0)\right\}g^*(m\delta) &= f(m\delta)\\ &\quad + \delta\left\{\frac{1}{2}W(m\delta)g(0) + \sum_{i=1}^{m-1} W([m-i]\delta)g^*(i\delta)\right\}\end{aligned} \tag{6}$$

where an asterisk denotes an approximate value (and we use the convention throughout that $\sum_{i=1}^{0} \equiv 0$).

Unless both $a = 0$ and $W(0) = 0$ we can successively evaluate

$$g^*(\delta),\ g^*(2\delta),\ \cdots.$$

If $a = 0$ and $W(0) = 0$, then the equivalent equation (4) (or if necessary an equation obtained by further differentiation) is solved by the method described. The additional errors involved in numerical differentiation (assuming the functions to have been given in tabulated form) are discussed in Section 8.

The equation corresponding to (6) involving the true values is

$$\begin{aligned}\left\{a - \frac{\delta}{2}W(0)\right\}g(m\delta) &= f(m\delta)\\ &\quad + \delta\left\{\frac{1}{2}W(m\delta)g(0) + \sum_{i=1}^{m-1} W([m-i]\delta)g(i\delta)\right\} + \sum_{i=1}^{m} e_{m,i}\end{aligned} \tag{7}$$

where

$$\begin{aligned}e_{m,i} &= \int_{(i-1)\delta}^{i\delta} W(m\delta - \xi)g(\xi)\,d\xi\\ &\quad - \frac{\delta}{2}\left\{W(m\delta - [i-1]\delta)g([i-1]\delta) + W(m\delta - i\delta)g(i\delta)\right\}.\end{aligned} \tag{8}$$

Writing

$$E_m = g^*(m\delta) - g(m\delta) \tag{9}$$

(i.e., E_m is the error after m steps) equations (6) and (7) give

$$\left\{a - \frac{\delta}{2}\,W(0)\right\} E_m = \delta \sum_{i=1}^{m-1} W([m-i]\delta)E_i - \sum_{i=1}^{m} e_{m,i}. \tag{10}$$

Now (see, for example, [5]) from (8)

$$e_{m,i} = -\frac{1}{12}\,\delta^2 \left\{\frac{d}{d\xi}\,[W(m\delta - \xi)g(\xi)]_{\xi=i\delta} - \frac{d}{d\xi}\,[W(m\delta - \xi)g(\xi)]_{\xi=(i-1)\delta}\right\} + O(\delta^5).$$

If we assume that g and W have bounded derivatives up to the second, it follows that $e_{m,i}$ is $O(\delta^3)$.

From equation (10) it follows by induction that a positive constant K can be found such that

$$|E_m| \leq f_m$$

where

$$\left\{a - \frac{\delta}{2}\,W(0)\right\} f_m = K\delta \sum_{i=1}^{m-1} f_i + mK\delta^3. \tag{11}$$

Rewriting (11) with $(m+1)$ for m and subtracting (11) from the result,

$$\left\{a - \frac{\delta}{2}\,W(0)\right\} f_{m+1} - \left\{a - \frac{\delta}{2}\,W(0) + \delta K\right\} f_m = K\delta^3.$$

The solution of this difference equation with initial condition from (11) is

$$f_m = C_1 \left\{\frac{a - \frac{\delta}{2}\,W(0) + \delta K}{a - \frac{\delta}{2}\,W(0)}\right\}^m - \delta^2 \tag{12}$$

where

$$C_1 = \frac{K\delta^3 + \delta^2\left(a - \frac{\delta}{2}\,W(0)\right)}{a - \frac{\delta}{2}\,W(0) + \delta K}.$$

The two cases $a \neq 0$ and $a = 0$ are discussed separately in the next two sections.

6. Equation of the Second Kind ($a \neq 0$). In this case (12) provides the required bound. It is similar to the bound obtained for the error in the numerical solution of an ordinary differential equation in [4].

At a fixed value of x we consider a sequence of values of δ such that $m\delta = x$ and $m \to \infty$ as $\delta \to 0$. Then for small δ the term in (12) with index m is bounded and $C_1 = 0(\delta^2)$. So for a fixed value of x, $f_m = 0(\delta^2)$, hence $E_m = O(\delta^2)$.

By considering the particular case $W = \text{constant} = K > 0$ and $g'' = \text{constant} = 12$ it can be seen that the error E_m can attain its bound f_m, and so, for given step size, it is possible for E_m to increase with m like A^m, where $A > 1$.

7. Equation of the First Kind ($a = 0$). We assume in this section that $W(0) \neq 0$, in which case equation (1), with $a = 0$, can be solved directly by means of the numerical formula (6).

In this case equation (12) only provides a convergent bound for E_m (at a fixed value of x) if

$$\left| 1 - \frac{2K}{W(0)} \right| \leq 1. \tag{13}$$

In general K cannot be chosen to satisfy this condition. An example of an equation for which (12) does provide a bound is

$$\int_0^x g(\xi)\, d\xi = 2x^3, \tag{14}$$

in which W = constant = 1. In this case it is easily shown that the errors satisfy the equation

$$-\frac{1}{2} E_m = \sum_{i=1}^{m-1} E_i + m\delta^2. \tag{15}$$

On comparison with (11) it can be seen that we can choose $K = 1$ in order to satisfy the equality $E_m = f_m$. In this case (13) is satisfied, and as in Section 6 it follows that for a fixed value of x, $E_m = O(\delta^2)$. A point of interest is that the errors E_m in this example form an oscillating sequence, viz., $-2\delta^2, 0, -2\delta^2, 0, \cdots$. This behavior is typical of the errors when $a = 0$, as is shown by the following analysis, which also shows that a convergent bound can always be found.

Equation (10) in the present case reduces to

$$-\frac{\delta}{2} W(0)\, E_m = \delta \sum_{i=1}^{m-1} W([m - i]\delta) E_i - \sum_{i=1}^{m} e_{m,i}. \tag{16}$$

So

$$-\frac{\delta}{2} W(0)\{E_{m+1} - E_m\} = \delta W(\delta) E_m + J_m \tag{17}$$

where

$$J_m = \delta \sum_{i=1}^{m-1} \{W([m + 1 - i]\delta) - W([m - i]\delta)\} E_i - e_{m+1,m+1} - \sum_{i=1}^{m} (e_{m+1,i} - e_{m,i}). \tag{18}$$

Equation (17) can be rearranged to give

$$-\frac{\delta}{2} W(0)\{E_{m+1} + E_m\} = \delta\{W(\delta) - W(0)\} E_m + J_m . \tag{19}$$

The right hand side of (19) is in general smaller than the right hand side of (17). That is, $E_{m+1} + E_m$ is in general smaller than $E_{m+1} - E_m$, which implies that E_{m+1} and E_m are in general of opposite sign. The E_m then forms an oscillating

sequence. A smaller bound than that given by equation (12), which is obtained essentially from (10) by replacing the errors by their moduli, can therefore be obtained by relating E_{m+2} to E_m.

Replacing m by $m + 1$ in (19) and subtracting (19) as it stands gives an equation for $E_{m+2} - E_m$. Assuming that g and W have bounded derivatives up to fourth order, it follows that the first and second differences of W are respectively of orders δ and δ^2 and that the first and second differences of $e_{m,i}$ are respectively of orders δ^4 and δ^5. It can then be shown inductively that a positive constant K can be found such that

$$(20) \qquad | E_m | \leqq h_m$$

where

$$(21) \qquad h_{m+2} - h_m = K\left\{\delta^2 \sum_{i=1}^{m-1} h_i + \delta(h_{m+1} + h_m) + m\delta^4 + \delta^3\right\},$$

and h_1, h_2 are chosen so that (20) holds for $m = 1, 2$.

From (16) it can be seen that h_1 and h_2 can be chosen so that

$$(22) \qquad h_1 = O(\delta^2), \qquad h_2 = O(\delta^2)$$

and

$$(23) \qquad h_1 < h_2 .$$

Then

$$(24) \qquad h_{2m-1} < h_{2m}, \quad \text{all } m$$

and hence it may be verified by induction that if

$$(25) \qquad H_{2m+2} - H_{2m} = K\left\{\delta^2\left(2\sum_{i=1}^{m-1} H_{2i} + H_{2m}\right) + \delta\left(H_{2m} + H_{2m+2}\right) + 2m\delta^4 + \delta^3\right\}$$

and

$$(26) \qquad H_2 = h_2$$

then

$$(27) \qquad h_{2m} < H_{2m} \quad \text{all } m > 1.$$

From (25) there is obtained the difference equation

$$(28) \qquad H_{2m+4}(1 - K\delta) - H_{2m+2}(2 + K\delta^2) + H_{2m}(1 - K\delta^2 + K\delta) = 2K\delta^4.$$

The solution of (28) is

$$(29) \qquad H_{2m} = -\delta^2 + C_1 p_1^{\,m} + C_2 p_2^{\,m}$$

where p_1, p_2 are the two values of

$$(30) \qquad \frac{1}{1 - K\delta}\left\{1 + \frac{K\delta^2}{2} \pm \left[K(2 + K)\delta^2 - K^2\delta^3 + \frac{K^2\delta^4}{4}\right]^{1/2}\right\}$$

and C_1, C_2 are given by the initial conditions

$$(31) \qquad \begin{cases} p_1 C_1 + p_2 C_2 = H_2 + \delta^2 \\ p_1^{\,2} C_1 + p_2^{\,2} C_2 = H_4 + \delta^2. \end{cases}$$

Now from (22) and (26), $H_2 = O(\delta^3)$ and from (25), $H_4 = H_2(1 + O(\delta))$. Also from (30), $p_1 = 1 + O(\delta)$, $p_2 = 1 + O(\delta)$, $p_2 - p_1 = O(\delta)$. So from (31),

$$C_1 = \frac{p_2(H_2 + \delta^3) - (H_4 + \delta^3)}{p_1(p_2 - p_1)} = \frac{O(\delta^3)}{O(\delta)} = O(\delta^2).$$

Similarly $C_2 = O(\delta^2)$.

At a fixed value of x we consider a sequence of values of δ such that $2m\delta = x$, thus $m \to \infty$ as $\delta \to 0$. Then for small δ the terms in (29) with index m are bounded and C_1, C_2 are $O(\delta^2)$. So for fixed x, H_{2m} is $O(\delta^2)$ and from (27), (24) and (20) it follows that $E_m = O(\delta^2)$.

8. Comparison of Methods. An equation of the first kind ($a = 0$) can be converted into an equation of the second kind by differentiation. For example, if $W(0) \neq 0$ equation (4) results. It has already been shown in Sections 6 and 7 that for both types of equation the truncation error at a given value of the independent variable is $O(\delta^2)$. In this section the effects of rounding errors in the two cases are briefly considered and the problem of whether to convert a first order equation into a second order one by differentiation before solving numerically is discussed.

In the case $a = 0$ the numerical formula (6) becomes

$$(32) \quad g^*(m\delta) = \frac{2}{W(0)}\left\{-\frac{f(m\delta)}{\delta} - \left[\frac{1}{2}W(m\delta)g(0) + \sum_{i=1}^{m-1} W([m - i]\delta)g^*(i\delta)\right]\right\}.$$

In the first place, as was shown in Section 7, the truncation error in $g^*(m\delta)$ obtained by using the formula (32) is in general of opposite sign for two consecutive values of m. Because of this the truncation error in the summation in the right hand side of (32) and hence in $g^*(m\delta)$ is smaller than it would be if the errors were of the same sign. To take advantage of this in the numerical computation, several digits must be retained beyond the point where the truncation error begins before rounding off. This implies that W must be known to several more significant figures than would otherwise be necessary. It is also evident on comparison of (6) and (32) that in the latter formula more digits in $f(m\delta)$ must be retained before rounding because of the δ in the denominator.

Suppose now that we are to solve an equation of the first kind, viz. (1) with $a = 0$. The functions f and W are supposed to have been given in tabular form and $W(0) \neq 0$. Then the equation can be solved directly or it can be differentiated first, giving (4), an equation of the second kind.

If the first method is adopted, then, as has been shown, the truncation errors will in general be of opposite sign and the rounding errors must be kept several digits smaller than the truncation error. If an automatic computer is being used it may not be inconvenient to retain these extra significant figures. However, f and W must be known to the extra degree of accuracy. Since the truncation errors are in general of opposite sign a simple smoothing process may be employed to improve the solution as follows. Denote the sequence of numbers

$$g(0),\ g^*(2\delta),\ g^*(4\delta),\ \cdots$$

by the symbols

$$g_e(0),\ g_e(2\delta),\ g_e(4\delta),\ \cdots$$

and complete the sequence

$$g_e(0),\ g_e(\delta),\ g_e(2\delta),\ \cdots$$

by interpolation.

Similarly, from the sequence of numbers

$$g^*(\delta),\ g^*(3\delta),\ g^*(5\delta),\ \cdots$$

the sequence

$$g_0(\delta),\ g_0(2\delta),\ g_0(3\delta),\ \cdots$$

is formed by using interpolation.

The two sequences $g_e(m\delta)$, $g_0(m\delta)$ then give approximate bounds to the solution, and the smoothed solution is given by $g_s(m\delta) = \frac{1}{2}[g_e(m\delta) + g_0(m\delta)]$. This procedure is illustrated in the numerical computation at the end of this section (Table 1).

If the second method is adopted, that is, equation (4) is solved instead, the functions f and W must first be differentiated numerically. Once this has been done the numerical solution can be rounded off to the same degree of accuracy as the truncation error and consequently fewer significant figures need be retained. However, extra significant figures in f and W now have to be used in the first place to obtain f' and W' to the required degree of accuracy when using numerical differentiation.

Each method appears to have its advantages, and the choice must depend on the data provided in a given problem, that is, on the spacing and number of significant figures in the given functions.

We conclude this section by presenting the details of a simple numerical computation in which the results given by the two methods can be compared.

The equation is

$$\int_0^x W(x - \xi)g(\xi)\,d\xi = f(x)$$

where $W(x) = \cos x$ and $f(x) = \sin x$. The analytical solution is $g(\xi) = 1$. The functions are given to 4 significant figures at intervals of 0.1 in x. The functions are differentiated numerically using a three-point formula except for the values at $x = 0$ where a four-point formula is used. The interpolation in the formation of the sequences $g_e(m\delta)$, $g_0(m\delta)$ is linear.

TABLE 1

x	$W(x)$ ($= \cos x$)	$f(x)$ ($= \sin x$)	$g^*(x)$ (direct solution)	$g_e(x)$	$g_0(x)$	$g_s(x)$	$W'(x)$ (3-point numerical formula)	$f'(x)$ (3-point formula)	$g^*(x)$ (differentiated equation)
0	1.0000	0	1.000	1.000	1.000	1.000	0	1.000	1.000
0.1	0.9950	0.0999	1.003	1.000	1.003	1.002	−0.100	0.994	0.999
0.2	0.9800	0.1988	1.000	1.000	1.000	1.000	−0.198	0.978	0.998
0.3	0.9554	0.2954	0.997	1.003	0.997	1.000	−0.294	0.953	0.997
0.4	0.9211	0.3894	1.006	1.006	0.998	1.002	−0.389	0.920	0.998
0.5	0.8776	0.4795	0.998	1.004	0.998	1.001	−0.479	0.876	0.998
0.6	0.8253	0.5647	1.003	1.003	0.996	1.000	−0.564	0.823	0.997
0.7	0.7649	0.6441	0.995	1.006	0.995	1.000	−0.642	0.763	0.997
0.8	0.6968	0.7173	1.008	1.008	0.994	1.001	−0.716	0.696	0.998
0.9	0.6216	0.7833	0.994	1.008	0.994	1.001	−0.783	0.621	0.998
1.0	0.5402	0.8415	1.009	1.009					

9. The System of Integral Equations. In this section we consider the numerical solution of the system (2). We use the expression $r(\mathbf{A})$ to denote the rank of $\mathbf{A}$.

If $r(\mathbf{A}) = n$, the step-by-step process is started by solving the algebraic set of equations

$$\mathbf{Ag}(0) = \mathbf{f}(0). \tag{33}$$

If $r(\mathbf{A}) < n$ then n linear combinations of the equations (2) can be chosen to give another system of the same form, with a matrix in which $n - r$ rows are identically zero. If the $n - r$ corresponding equations are differentiated, the rank of the matrix of the resulting set of n equations will in general have increased (in the same way that differentiation of the equation of the first kind, viz. (1) with $a = 0$, in general gives equation (4) of the second kind). We assume that repeated application of this process eventually leads to a system of the same form as (2) with a matrix of rank n (this corresponds to the assumption in Section 5: $W^{(n)}(0) \neq 0$ for some n). Setting $x = 0$ in this new system, we obtain a set of equations analogous to (33) with which to start the step-by-step process.

Equal intervals of length δ are now taken along the x-axis. For the mth step we have

$$\mathbf{Ag}(m\delta) - \int_0^{m\delta} \mathbf{W}(m\delta - \xi)\mathbf{g}(\xi)\, d\xi = \mathbf{f}(m\delta).$$

Replacing the integral by the corresponding trapezoidal rule approximation and rearranging:

$$\begin{aligned} \left\{\mathbf{A} - \frac{\delta}{2}\mathbf{W}(0)\right\}\mathbf{g}^*(m\delta) &= \mathbf{f}(m\delta) \\ &\quad + \delta\left\{\frac{1}{2}\mathbf{W}(m\delta)\mathbf{g}(0) + \sum_{i=1}^{m-1}\mathbf{W}([m-i]\delta)\mathbf{g}^*(i\delta)\right\} \\ &= \mathbf{Q}(m\delta) \quad \text{(say)}, \end{aligned} \tag{34}$$

where an asterisk denotes an approximate value.

Provided $\left|\mathbf{A} - \frac{\delta}{2}\mathbf{W}(0)\right| \neq 0$ we can successively evaluate

$$\mathbf{g}^*(\delta), \mathbf{g}^*(2\delta), \cdots$$

where at each step we solve a set of n equations (with coefficients independent of n).

The above method is exactly analogous to that described for the single integral equation in Section 5. Only the outlines of proofs of convergence are given in this section, the details being analogous to those given for the single integral equation. The equation for the truncation error corresponding to (10) is

$$\left\{\mathbf{A} - \frac{\delta}{2}\mathbf{W}(0)\right\}\mathbf{E}_m = \delta\sum_{i=1}^{m-1}\mathbf{W}([m-i]\delta)\mathbf{E}_i - \sum_{i=1}^{m}\mathbf{e}_{m,i} \tag{35}$$

where

$$\mathbf{E}_m = \mathbf{g}^*(m\delta) - \mathbf{g}(m\delta)$$

and

$$\mathbf{e}_{m,i} = \int_{(i-1)\delta}^{i\delta} \mathbf{W}(m\delta - \xi)\mathbf{g}(\xi)\, d\xi$$
$$- \frac{\delta}{2}\{\mathbf{W}(m\delta - [i-1]\delta)\mathbf{g}([i-1]\delta) + \mathbf{W}(m\delta - i\delta)\mathbf{g}(i\delta)\}.$$

The case of $r(\mathbf{A}) = n$ corresponds to the case $a \neq 0$ in Section 5 and a quantity f_m can be found which is a bound for every component of $\mathbf{E}_m$ and is of the same form as (12) (with $a \neq 0$). So when $r(\mathbf{A}) = n$, at a fixed value of x we have $\mathbf{E}_m = \mathbf{O}(\delta^2)$. In this case there is no condition on $r(\mathbf{W}(0))$.

If $r(\mathbf{A}) < n$ the analysis proceeds as in Section 7. In this case we assume, to avoid further complication, $r(\mathbf{W}(0)) = n$. The equation corresponding to (17) is

$$\left\{\mathbf{A} - \frac{\delta}{2}\mathbf{W}(0)\right\}\{\mathbf{E}_{m+1} - \mathbf{E}_m\} = \delta\mathbf{W}(\delta)\mathbf{E}_m + \mathbf{J}_m \tag{36}$$

where

$$\begin{aligned}\mathbf{J}_m = \delta \sum_{i=1}^{m-1} \{\mathbf{W}([m+1-i]\delta) - \mathbf{W}([m-i]\delta)\}\mathbf{E}_i \\ - \mathbf{e}_{m+1,m+1} - \sum_{i=1}^{m} (\mathbf{e}_{m+1,i} - \mathbf{e}_{m,i}).\end{aligned} \tag{37}$$

Equation (36) can be rearranged to give

$$\left\{\mathbf{A} - \frac{\delta}{2}\mathbf{W}(0)\right\}\{\mathbf{E}_{m+1} + \mathbf{E}_m\} = 2\mathbf{A}\mathbf{E}_m + \delta\{\mathbf{W}(\delta) - \mathbf{W}(0)\}\mathbf{E}_m + \mathbf{J}_m. \tag{38}$$

This is of the same form as equation (19) except for the term $2\mathbf{AE}_m$. It is now shown that when $r(\mathbf{A}) < n$ this term only contributes terms to the elements of $\mathbf{E}_{m+1} + \mathbf{E}_m$ of the same order as the contribution of the remaining terms on the right hand side of (38). We assume that the elements of the last $n - r$ rows of $\mathbf{A}$ are all zero (this can be arranged by taking linear combinations of the original equations). Then

$$\left|\mathbf{A} - \frac{\delta}{2}\mathbf{W}(0)\right| = O(\delta^{n-r}),$$

and it is easily verified that each element of $\mathbf{E}_{m+1} + \mathbf{E}_m$ consists of a linear combination of the elements of the right hand side of (38), the coefficients of the first r elements being $O(1)$ and those of the last $n - r$ elements being $O(\delta^{-1})$. But all the last $n - r$ elements of $\mathbf{AE}_m$ vanish, so the dominant terms in each element of $\mathbf{E}_{m+1} + \mathbf{E}_m$ are of the type

(a) $O(\delta^{-1})\{\delta[\mathbf{W}(\delta) - \mathbf{W}(0)]\mathbf{E}_m + \mathbf{J}_m\}_i$

where $r + 1 \leqq i \leqq n$,

(b) $O(1)\{\mathbf{AE}_m\}_i$

where $1 \leqq i \leqq r$.

The term (a) is analogous to the expression for $E_{m+1} + E_m$ from equation (19).

From equation (35)

$$\mathbf{A}\mathbf{E}_m = \frac{\delta}{2}\mathbf{W}(0)\mathbf{E}_m + \delta \sum_{i=1}^{m-1} \mathbf{W}([m - i]\delta)\mathbf{E}_i - \sum_{i=1}^{m} \mathbf{e}_{m,i}.$$

It can be seen that contributions from (a) and (b) are of the same order.

The situation is the same as in Section 7, the elements of $\mathbf{E}_{m+1} + \mathbf{E}_m$ being in general smaller than those of $\mathbf{E}_{m+1} - \mathbf{E}_m$, consecutive errors thus being in general of opposite sign. Replacing m by $m + 1$ in (38) and subtracting (38) as it stands gives an equation for $\mathbf{E}_{m+2} - \mathbf{E}_m$. Assuming that the elements of $\mathbf{g}$ and $\mathbf{w}$ have bounded derivatives up to the fourth order it can be shown inductively that a positive constant K can be found such that

$$|\{\mathbf{E}_m\}_i| \leqq h_m, \quad \text{for all } i$$

where h_m is given by (21), and (22), (23) hold. The analysis of Section 7 then shows that for a fixed value of x we have $\mathbf{E}_m = \mathbf{O}(\delta^2)$.

As indicated at the beginning of this section, if $r(\mathbf{A}) < n$ the system can in general be converted into an equivalent system with $r = n$ by application of the operations of addition and differentiation. The problem of whether to convert a system with $r < n$ into an equivalent system in this way, before solving numerically, is exactly analogous to that discussed in Section 8, and will not be treated further here. The merits of the two methods are summarized in the conclusions.

10. Conclusions. The numerical solution of a convolution integral equation of the Volterra type, equation (1), by using a simple quadrature formula has been discussed. It is shown that if the step length is δ, the truncation error at a fixed value of the independent variable is $O(\delta^2)$ both for equations of the first and second kinds. However, the behavior of the truncation error is different in the two cases, being in general of opposite sign for consecutive steps in the case of an equation of the first kind. Since an equation of the first kind can in general be converted into one of the second kind by differentiation, it can either be solved numerically as it stands or differentiated first. The merits of the two methods appear to be as follows. In the direct method, both a smooth solution and approximate bounds for the truncation error can be obtained simultaneously. However, more significant figures have to be retained throughout before rounding off. If the equation is differentiated first, the given functions (which are assumed to be given in tabulated form) have to be differentiated numerically. In the actual step-by-step computation fewer significant figures need be retained. If the given functions are tabulated to sufficient significant figures to permit accurate derivatives to be obtained using numerical differentiation, the second method will probably be the more accurate. Otherwise, and if it is convenient to retain the extra significant figures throughout, the direct method has the advantages of requiring fewer steps (no numerical differentiation) and of providing approximate bounds for the truncation error. The method has been generalized to include the numerical solution of the system of integral equations (2). An instance of a practical problem arising from supersonic linearized theory in which this equation occurs has been described. The truncation error at a fixed value of the independent variable is again $O(\delta^2)$. The behavior of the truncation error is analogous to that in the solution of the single integral equation, the case $r(\mathbf{A}) < n$

corresponding to an equation of the first kind, and the case $r(\mathbf{A}) = n$ corresponding to an equation of the second kind.

11. Acknowledgement. The author is indebted to the referee for some helpful suggestions.

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Numerical Integration Formulas of Degree 3 for Product Regions and Cones

By A. H. Stroud

1. Introduction. Here we discuss approximate integration formulas of degree 3; that is, formulas which are exact for polynomials of degree $\leqq 3$, of the form

$$\int_R f(x_1, \cdots, x_n)\, dx_n \cdots dx_1 \simeq \sum b_i f(v_{i1}, \cdots, v_{in})$$

for certain regions R in n-dimensional, real, Euclidean space E_n.

If R is the Cartesian product of an r-dimensional and an s-dimensional region, $R = R_r \times R_s$, $r + s = n$, and if formulas of degree k involving p and q points are known for R_r and R_s, then Hammer and Wymore [4] have shown that a formula of degree k may be obtained for R which involves pq points. Similarly, if R is a cone (or pyramid) with $(n - 1)$-dimensional base B, and if a formula of degree k involving p points is known for B, and further, if a formula of degree k of the form

$$\int_0^1 x^{n-1} g(x)\, dx \simeq \sum a_i g(v_i)$$

is known involving q points, then Hammer, Marlowe and Stroud [2] have shown that a formula of degree k can be obtained for R which involves pq points.

For $k = 3$ we will show that these results can be improved as follows. If $R = R_r \times R_s$ and formulas of degree 3 involving p and q points are known for R_r and R_s, then a formula of degree 3 can be found for R involving $p + q + 1$ points (and in some cases this may be reduced to $p + q$ or $p + q - 1$ points). If R is a cone and a formula of degree 3 involving p points is known for its base B, then a formula of degree 3 involving $p + 3$ points (in some cases $p + 2$ or $p + 1$ points) can be found for R, whereas the method of [2] would involve $2p$ points. We also give a $p + 2$ point formula of degree 3 for certain double cones, where p is the number of points in a formula for the base, and a $2n + 3$ point formula of degree 3 for an n-dimensional simplex.

2. Formulas for Product Regions. For convenience let us assume that the centroid of R_r is at the origin of coordinates in the subspace E_r of E_n for which $x_{r+1} = \cdots = x_n = 0$, and also that the centroid of R_s is at the origin in the subspace E_s for which $x_1 = \cdots = x_r = 0$. Then the centroid of R is at the origin in E_n. Also, for convenience in notation, we will write

$$R_r(x_1^{\alpha_1} \cdots x_r^{\alpha_r}) \equiv \int_{R_r} x_1^{\alpha_1} \cdots x_r^{\alpha_r}\, dx_r \cdots dx_1$$

with similar notations for R_s and R. Then, for example, $R_r(1)$ is the r-dimensional content or volume of R_r and $R(1) = R_r(1)R_s(1)$ is the n-dimensional volume of R.

Received January 18, 1960. This paper was supported by the Graduate Research Committee of the University of Wisconsin.

Suppose an approximate integration formula of degree 3 for R_r is given by:

$$\nu_i = (\nu_{i1}, \cdots, \nu_{ir}) \qquad a_i, \qquad i = 1, 2, \cdots, p. \tag{1}$$

Also suppose that a formula of degree 3 for R_s is given by:

$$\nu_{p+j} = (\nu_{p+j,r+1}, \cdots, \nu_{p+j,n}) \qquad a_{p+j}, \qquad j = 1, 2, \cdots, q. \tag{2}$$

At first it is assumed no ν_i coincides with the origin in E_r and no ν_{p+j} coincides with the origin in E_s. An integration formula for degree 3 for R is given by:

$$\begin{aligned} \nu_0 &= (0, \cdots, 0, 0, \cdots, 0) & b_0 &= -R(1) \\ \nu_i &= (\nu_{i1}, \cdots, \nu_{ir}, 0, \cdots, 0) & b_i &= a_i R_s(1), \\ & & & \qquad i = 1, 2, \cdots, p \\ \nu_{p+j} &= (0, \cdots, 0, \nu_{p+j,r+1}, \cdots, \nu_{p+j,n}) & b_{p+j} &= a_{p+j} R_r(1), \\ & & & \qquad j = 1, 2, \cdots, q. \end{aligned} \tag{3}$$

To prove this statement, it suffices to prove that the formula is exact for the following ten types of monomials: $1, x_{r_1}, x_{s_1}, x_{r_1}x_{r_2}, x_{r_1}x_{s_1}, x_{s_1}x_{s_2}, x_{r_1}x_{r_2}x_{r_3}, x_{r_1}x_{r_2}x_{s_1}, x_{r_1}x_{s_1}x_{s_2}, x_{s_1}x_{s_2}x_{s_3}$ where $r_1, r_2, r_3 = 1, 2, \cdots, r$ and $s_1, s_2, s_3 = r+1, r+2, \cdots, n$:

The monomial 1. Since $a_1 + \cdots + a_p = R_r(1)$ and $a_{p+1} + \cdots + a_{p+q} = R_s(1)$ we have

$$-R(1) + [a_1 + \cdots + a_p]R_s(1) + [a_{p+1} + \cdots + a_{p+q}]R_r(1) = -R(1) + 2R_r(1)R_s(1) = R(1).$$

The type $x_{r_1}x_{r_2}x_{r_3}$. The formula gives for the integral of $x_{r_1}x_{r_2}x_{r_3}$ over R

$$[a_1\nu_{1r_1}\nu_{1r_2}\nu_{1r_3} + \cdots + a_p\nu_{pr_1}\nu_{pr_2}\nu_{pr_3}]R_s(1) = R_r(x_{r_1}x_{r_2}x_{r_3})R_s(1).$$

Thus the formula is exact because

$$R(x_{r_1}x_{r_2}x_{r_3}) = R_r(x_{r_1}x_{r_2}x_{r_3})R_s(1).$$

The proofs for types $x_{r_1}, x_{s_1}, x_{r_1}x_{r_2}, x_{s_1}x_{s_2}, x_{s_1}x_{s_2}x_{s_3}$ are similar.

The type $x_{r_1}x_{r_2}x_{s_1}$. By the formula the integral of $x_{r_1}x_{r_2}x_{s_1}$ over R is zero since each term of the sum is zero. However

$$R(x_{r_1}x_{r_2}x_{s_1}) = R_r(x_{r_1}x_{r_2})R_s(x_{s_1}) = 0$$

because $R_s(x_{s_1}) = 0$. The proofs for the types $x_{r_1}x_{s_1}$ and $x_{r_1}x_{s_1}x_{s_2}$ are similar.

If the formulas for R_r and R_s already include the origin, then the formula for R will involve either $p + q$ or $p + q - 1$ points according as the origin is included in one or both of these formulas. In the former case, if ν_{p+1} in (2) is the origin, then $b_0 = -[a_{p+2} + \cdots + a_{p+q}]R_r(1)$. In the latter case, if both ν_1 in (1) and ν_{p+1} in (2) are the origin, then

$$b_0 = R(1) - [a_2 + \cdots + a_p]R_s(1) - [a_{p+2} + \cdots + a_{p+q}]R_r(1).$$

Depending on the particular structure of the formulas (1) and (2) it may be possible in special cases to eliminate the origin in the formula (3) for R. Suppose in the formula (2) for R_s, ν_{p+1} is the origin and a_{p+1} is positive. Also suppose R_r is centrally symmetric. Then, as shown in [6], we may obtain $2r$ point formulas for

R_r with $-\nu_{r+i} = \nu_i$, $a_{r+i} = a_i$, $i = 1, 2, \cdots, r$. In addition to the assumptions already made about R_r, we may also assume $R_r(x_1^2) = \cdots = R_r(x_r^2)$ and $R_r(x_ix_j) = 0$, for $i \neq j$; then in order that (1) be a formula of degree 3 it is necessary and sufficient that $\nu_1, \cdots, \nu_r$ be orthogonal vectors with $a_i = \dfrac{R_r(x_1^2)}{2|\nu_i|^2}$ where $|\nu_i|$ is the distance of ν_i from the origin. Since in the formula for $R_r \times R_s$ it is no longer necessary that $b_1 + \cdots + b_{2r} = R(1)$, a formula of degree 3 for $R_r \times R_s$ is given by the $2r + q - 1$ points:

$$-\nu_{r+i} = \nu_i = (\nu_{i1}, \cdots, \nu_{ir}, 0, \cdots, 0) \qquad b_{r+i} = b_i, \quad i = 1, 2, \cdots, r$$

$$\nu_{2r+j} = (0, \cdots, 0, \nu_{2r+j,r+1}, \cdots, \nu_{2r+j,n}) \qquad b_{2r+j} = a_{2r+j}R_r(1), \quad j = 2, \cdots, q$$

where the only conditions that the b_i and ν_i, $i = 1, 2, \cdots, r$, must satisfy are $2(b_1 + \cdots + b_r) = a_{2r+1}R_r(1)$ and $\nu_1, \cdots, \nu_r$ are any set of orthogonal vectors for which $|\nu_i|^2 = \dfrac{R_r(x_1^2)}{2b_i}$; the b_i must all be positive.

Of course if both R_r and R_s are centrally symmetric then $R_r \times R_s$ is also centrally symmetric and the results of [6] may be applied directly to obtain $2n$ point formulas. Some of the resulting formulas may also be obtained from the separate formulas for R_r and R_s by a method somewhat similar to that described in the preceding paragraph.

As an example of a specific formula we give a formula for the region $C_r \times S_{n-r}$, $1 \leqq r \leqq n - 2$, where C_r is the r-cube with vertices $(\pm 1, \pm 1, \cdots, \pm 1)$ and S_{n-r} is the $(n - r)$-simplex with vertices $(-1, -1, \cdots, -1)$, $(1, 0, \cdots, 0)$, $(0, 1, \cdots, 0), \cdots, (0, 0, \cdots, 1)$. For $r = 1$, $C_1 \times S_{n-1}$ can be considered as a prism (or in the terminology of Sommerville [5], p. 113, a prism of the first species) with base S_{n-1}; for general r, $C_r \times S_{n-r}$ is a prism of species r. Since C_r is centrally symmetric the remarks made above apply. A formula of degree 3 is known for S_{n-r} using $n - r + 2$ points, [3], one of which is the origin, but the origin is taken with a negative weight so that it is necessary to use $n + r + 2$ points in the formula for $C_r \times S_{n-r}$. A particular formula is:

$$\nu_0 = (0, 0, \cdots, 0, 0, 0, \cdots, 0) \qquad b_0 = 2^r S_{n-r}(1)\left[1 - \frac{r}{3} - \frac{(n-r+3)^2}{4(n-r+2)}\right]$$

$$-\nu_{r+1} = \nu_1 = (1, 0, \cdots, 0, 0, 0, \cdots, 0) \qquad b_{r+1} = b_1 = \cdots$$

$$-\nu_{r+2} = \nu_2 = (0, 1, \cdots, 0, 0\ 0, \cdots, 0) \qquad = b_{2r} = b_r = \frac{2^{r-1}}{3} S_{n-r}(1)$$

$$\cdots$$

$$-\nu_{2r} = \nu_r = (0, 0, \cdots, 1, 0, 0, \cdots, 0)$$

$$\nu_{2r+1} = \left(0, 0, \cdots, 0, \frac{-2}{n-r+3},\right.$$

$$\frac{-2}{n-r+3}, \cdots, \frac{-2}{n-r+3}\Big) \qquad b_{2r+1} = \cdots = b_{n+r+1} =$$

$$\nu_{2r+2} = \left(0, 0, \cdots, 0, \frac{2}{n-r+3}, 0, \cdots, 0\right) = \frac{(n-r+3)^2\, 2^r S_{n-r}(1)}{4(n-r+1)(n-r+2)}$$

$$\cdots$$

$$\nu_{n+r+1} = \left(0, 0, \cdots, 0, 0, 0, \cdots, \frac{2}{n-r+3}\right)$$

3. Formulas for Cones. Suppose the centroid of the $(n-1)$-dimensional base B is at the origin in the subspace E_{n-1} of E_n for which $x_1 = 0$ and that the vertex of R is at $(1, 0, \cdots, 0)$ in E_n. Suppose further a formula of degree 3 for B is given by

$$(4) \qquad \nu_i = (\nu_{i2}, \nu_{i3}, \cdots, \nu_{in}) \qquad a_i, \qquad i = 1, 2, \cdots, p.$$

We assume at first that no ν_i coincides with the origin in E_{n-1}. An integration formula of degree 3 for R can be found of the form

$$(5) \qquad \begin{aligned} \nu_i &= (\eta, \gamma\nu_{i2}, \gamma\nu_{i3}, \cdots, \gamma\nu_{in}) & b_i &= \alpha a_i, & i &= 1, 2, \cdots, p \\ \nu_{p+j} &= (\xi_j, 0, 0, \cdots, 0) & b_j&, & j &= 1, 2, 3. \end{aligned}$$

To prove this we must first calculate the monomial integrals over R in terms of those over B. These are

$$\int_R x_1^{\beta} x_2^{\beta_2} \cdots x_n^{\beta_n}\, dx_n \cdots dx_1 = \int_0^1 (1-x_1)^{n+\beta_2+\cdots+\beta_n-1} x_1^{\beta}\, dx_1 \int_B x_2^{\beta_2} \cdots x_n^{\beta_n}\, dx_n \cdots dx_2 .$$

The equations which must be satisfied by (5) if it is to integrate each monomial are given below (where the monomial from which the equation arises is indicated at the left):

$$[x_1^{\beta}] \qquad b_1\xi_1^{\beta} + b_2\xi_2^{\beta} + b_3\xi_3^{\beta} + \alpha\eta^{\beta}[a_1 + \cdots + a_p] = \frac{\beta!(n-1)!}{(n+\beta)!} B(1), \qquad \beta = 0, 1, 2, 3.$$

$$[x_1^{\beta} x_i] \qquad \alpha\gamma\eta^{\beta}[a_1\nu_{1i} + \cdots + a_p\nu_{pi}] = \frac{\beta!\,n!}{(n+\beta+1)!} B(x_i), \qquad \beta = 0, 1, 2$$

$$[x_1^{\beta} x_i x_j] \qquad \alpha\gamma^2\eta^{\beta}[a_1\nu_{1i}\nu_{1j} + \cdots + a_p\nu_{pi}\nu_{pj}] = \frac{\beta!(n+1)!}{(n+\beta+2)!} B(x_i x_j), \qquad \beta = 0, 1$$

$$[x_i x_j x_k] \qquad \alpha\gamma^3[a_1\nu_{1i}\nu_{1j}\nu_{1k} + \cdots + a_p\nu_{pi}\nu_{pj}\nu_{pk}] = \frac{1}{n+3} B(x_i x_j x_k)$$

where $i, j, k = 2, 3, \cdots, n$. Because we have assumed $B(x_i) = 0$, the three equations $[x_1^{\beta} x_i]$ are of the form $0 = 0$. From the equations $[x_i x_j]$ and $[x_1 x_i x_j]$ it imme-

diately results that $\eta = 1/(n+3)$. From equations $[x_i x_j]$ and $[x_i x_j x_k]$ it follows that

$$\alpha = \frac{(n+3)^2}{(n+2)^3}, \qquad \gamma = \frac{n+2}{n+3}.$$

Since $a_1 + \cdots + a_p = B(1)$ the equations that remain become

$$[x_1^\beta] \qquad b_1\xi_1^\beta + b_2\xi_2^\beta + b_3\xi_3^\beta + \frac{(n+3)^2}{(n+2)^3}\frac{1}{(n+3)^\beta}B(1) = \frac{\beta!(n-1)!}{(n+\beta)!}B(1),$$

$$\beta = 0, 1, 2, 3.$$

In these equations we cannot take, say, $b_3 = 0$, which would in effect reduce by one the number of points in the formula, because the resulting equations do not have a real solution. It might be expected, however, that there would be many solutions with all $b_i \neq 0$. Here we give just one of the simplest solutions. If we choose

$$b_3 = -\frac{(n+3)^2}{(n+2)^3}B(1), \qquad \xi_3 = \frac{1}{n+3}$$

then the other values are

$$b_1 = b_1'B(1) = \frac{2(n+1) - (n-1)\sqrt{2(n+1)(n+2)}}{4n(n+1)^2}B(1)$$

$$\xi_1 = \frac{2(n+2) + \sqrt{2(n+1)(n+2)}}{(n+2)(n+3)}$$

$$b_2 = b_2'B(1) = \frac{2(n+1) + (n-1)\sqrt{2(n+1)(n+2)}}{4n(n+1)^2}B(1)$$

$$\xi_2 = \frac{2(n+2) - \sqrt{2(n+1)(n+2)}}{(n+2)(n+3)}.$$

These values of b_1', b_2', ξ_1, ξ_2 have been given in connection with numerical integration with respect to a weight function x^{n-1} by Fishman [1] for $n = 1(1)6$ to 12D and in [2] for $n = 2(1)4$ to 18S. These authors have used n where we use $n - 1$ and for $m = 2$, $j = 1, 2$, their $1 - x_j$ is our ξ_j, their b_j is our b_j'. It is proved in [2] that the $2p$ points

$$(\xi_j, (1-\xi_j)\nu_{i2}, (1-\xi_j)\nu_{i3}, \cdots, (1-\xi_j)\nu_{in}) \qquad b_j'a_i$$

$i = 1, 2, \cdots, p$, $j = 1, 2$, are an integration formula of degree 3 for R.

If one of the ν_i in (4), say ν_1, is the origin in E_{n-1} then it may be possible to determine b_1, b_2, ξ_1, ξ_2 with $b_3 = 0$ to give a formula which involves only $p + 1$ points $\nu_2, \cdots, \nu_{p+2}$. The formula using $n + 2$ points when R is an n-simplex, [3], can be derived in this manner; the formula for B, which is an $(n-1)$-simplex, involves $n + 1$ points of which one is the origin in E_{n-1}. In any event, if ν_1 is the origin, we may derive a formula of degree 3 for R involving $p + 2$ points $\nu_2, \cdots, \nu_{p+3}$ where

$$b_3 = -\frac{(n+3)^2}{(n+2)^3}[a_2 + \cdots + a_p]$$

and the other values as before. Usually there will be other $p + 2$ point formulas as well.

Now we briefly discuss formulas for regions which are double cones; that is, regions which are the union of two cones with vertices $(1, 0, \cdots, 0)$ and $(-1, 0, \cdots, 0)$ and with a common base B of the type we have just considered. By a combination of the methods of this section and the preceding it is easy to see that from a formula (4) for B we may obtain the following formula involving $p + 2$ points for the double cone R:

$$\nu_i = (0, \gamma\nu_{i2}, \gamma\nu_{i3}, \cdots, \gamma\nu_{in}) \qquad b_i = \frac{2(n+3)^2}{(n+2)^3} a_i, \qquad i = 1, 2, \cdots, p$$

$$-\nu_{p+2} = \nu_{p+1} = (\xi, 0, 0, \cdots, 0) \qquad b_{p+2} = b_{p+1} = \frac{3n+8}{n(n+2)^3} B(1)$$

where

$$\gamma = \frac{n+2}{n+3} \quad \text{and} \quad \xi = \sqrt{\frac{2n^2+8n+8}{3n^2+11n+8}}.$$

Since $\xi < 1$ for all n, the points ν_{p+1} and ν_{p+2} are always interior to R. In this formula the origin is not required if it does not occur in (4) and b_{p+1} and ξ are uniquely determined. If the origin does occur in (4), then in some cases it can be eliminated in the formula for R; if ν_1 is the origin, then b_{p+1}, ξ, and b_1 will not be uniquely determined and we may take ξ arbitrary, which will then determine

$$b_{p+1} = \frac{2}{\xi^2 n(n+1)(n+2)} B(1)$$

$$b_1 = \frac{2}{n} B(1) - 2b_{p+1} - \frac{2(n+3)^2}{(n+2)^3} [a_2 + \cdots + a_p].$$

Since b_{p+1} is required to be positive, we could, providing

$$[a_2 + \cdots + a_p] < \frac{(n+2)^3}{n(n+3)^2} B(1),$$

or, in other words, providing

$$a_1 > -\frac{(3n+8)}{n(n+3)^2} B(1),$$

choose $b_1 = 0$. In this case the formula for R would involve only the $p + 1$ points $\nu_2, \cdots, \nu_{p+2}$ and b_{p+1} and ξ would again be uniquely determined. In this paragraph R has been a double cone of the first species with base B; this method may be repeated to give a formula for a double cone of species r. If we take a double cone of species r with base S_{n-r}, $1 \leqq r \leqq n - 2$, then this region has a formula of degree 3 which involves $n + r + 2$ points.

4. A Special Formula for Simplexes. We digress here from the methods of the previous sections to give a special formula of degree 3 for a simplex S_n in E_n. The formula involves $2n + 3$ points, $n \geqq 2$, all but one of which are on the surface of S_n; the formula involves the $n + 1$ vertices, the $n + 1$ centroids of the $(n - 1)$-dimensional faces, and the centroid of S_n.

To develop the formula it is most convenient to take S_n to have vertices $(0, 0, \cdots, 0)$, $(1, 0, \cdots, 0)$, $\cdots$, $(0, 0, \cdots, 1)$. Then the monomial integrals over S_n are

$$\int_{S_n} x_i^{\alpha_i} x_j^{\alpha_j} x_k^{\alpha_k}\, dx_n \cdots dx_1 = \frac{\alpha_i!\,\alpha_j!\,\alpha_k!}{(n + \alpha_i + \alpha_j + \alpha_k)!}.$$

The formula is then:

$$\begin{aligned}
\nu_0 &= \left(\frac{1}{n+1}, \frac{1}{n+1}, \cdots, \frac{1}{n+1}\right) & b_0 &= -\frac{(n+1)^2(n-3)}{(n+2)(n+3)} S_n(1) \\
\nu_1 &= (0, 0, \cdots, 0) & b_1 &= \cdots = b_{n+1} = \\
\nu_2 &= (1, 0, \cdots, 0) & &= \frac{3}{(n+1)(n+2)(n+3)} S_n(1) \\
&\cdots & & \\
\nu_{n+1} &= (0, 0, \cdots, 1) & & \\
\nu_{n+2} &= \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right) & b_{n+2} &= \cdots = b_{2n+2} = \\
\nu_{n+3} &= \left(0, \frac{1}{n}, \cdots, \frac{1}{n}\right) & &= \frac{n^3}{(n+1)(n+2)(n+3)} S_n(1) \\
&\cdots & & \\
\nu_{2n+2} &= \left(\frac{1}{n}, \frac{1}{n}, \cdots, 0\right) & &
\end{aligned}$$

where $S_n(1) = 1/(n!)$ is the volume of S_n. Because of the symmetries of this particular simplex, the proof that this is a formula of degree 3 can be established by verifying that it is exact for the 7 monomials: $1, x_1, x_1^2, x_1x_2, x_1^3, x_1^2x_2, x_1x_2x_3$. For $n = 1$ this formula gives us Simpson's formula, since for S_1, a line segment, the vertices coincide with the $(n-1)$-faces; the weight for each end point of S_1 is $b_1 + b_4 = b_2 + b_3$. For $n = 3$, $b_0 = 0$, and in this one case the centroid does not occur in the formula; for $n > 3$, b_0 is negative.

This formula will be useful when it is desired to integrate over a region by subdividing the region into simplexes and then applying a formula of degree 3 to each simplex. For a large number of subdivisions the total number of points used in applying this formula to each simplex will be less than the total number of points used if the $n + 2$ point formula is applied to each simplex. For example, for $n = 3$ suppose we desire to integrate over an icosahedron (which has 20 triangular faces) by subdividing it into 20 tetrahedra (each tetrahedron having as vertices the vertices of a face plus the center of the icosahedron). Repeated use of the formula given here would involve a total of 63 points whereas use of the 5-point formula for each tetrahedron would involve 100 points.

5. Concluding Remarks. From the formulas we have discussed, formulas may be obtained for regions which are linear transforms of the particular regions we have considered. While these results add greatly to our knowledge of formulas of degree 3

nothing is yet known concerning such formulas for regions which are less regular than those considered here.

It seems certain, although no proof is known, that the $n + 2$ point formula of degree 3 for S_n and the $2n$ point formulas of degree 3 for C_n involve the minimal number of points for this degree. If this is true, then it is also likely that the n + $r + 2$ point formulas for both $C_r \times S_{n-r}$ and the double cone of species r with base S_{n-r} are also minimal. For fixed r these latter regions are the duals of each other (see [5], p. 56) and thus have the same symmetries. This seems to indicate that the minimal point formulas of degree 3 for a region are related to the group of symmetries of the region.

I am indebted to R. G. Hetherington for many discussions on this subject.

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The Epsilon Algorithm and Operational Formulas of Numerical Analysis

By P. Wynn

1. Introduction. It is the purpose of this paper to describe a non-linear technique which appears to have powerful and general application in numerical analysis. However, before doing so it is necessary to refer to a few related theoretical concepts.

2. Rational Operational Formulas. The double sequence of rational functions

$$\frac{U_{\mu,\nu}(x)}{V_{\mu,\nu}(x)} \qquad \mu, \nu = 0, 1, \cdots$$

where

$$(1) \qquad \frac{U_{\mu,\nu}(x)}{V_{\mu,\nu}(x)} = \frac{\alpha_{\mu,\nu,0} + \alpha_{\mu,\nu,1}\, x + \cdots + \alpha_{\mu,\nu,\nu}\, x^{\nu}}{\beta_{\mu,\nu,0} + \beta_{\mu,\nu,1}\, x + \cdots + \beta_{\mu,\nu,\mu}\, x^{\mu}}$$

may be derived from the series

$$(2) \qquad \beta(x) = \sum_{s=0}^{\infty} c_s x^s$$

by imposing the condition that the power series expansion of (1) should agree with (2) as far as the term in $x^{\mu+\nu}$. If none of the Hankel determinants

$$\begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+k-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+k} \\ \vdots & \vdots & & \vdots \\ c_{m+k-1} & c_{m+k} & \cdots & c_{m+2k-2} \end{vmatrix} \qquad m, k-1 = 0, 1, \cdots$$

vanish, and the additional condition $\beta_{\mu,\nu,0} = 1$ is imposed, the coefficients in the rational expression (1) are uniquely determined. The rational expressions (1) may be placed in a two-dimensional array in which the quotient (1) occurs at the intersection of the $(\mu + 1)$th row and the $(\nu + 1)$th column. [1] [2] [3].

As is well known, the numerical convergence of the sequence

$$(3) \qquad \frac{U_{r,r}(x)}{V_{r,r}(x)} \qquad r = 0, 1, \cdots$$

for a particular value of x is in many cases much better than that of the series (2). This consideration led Kopal [4] to the consideration of rational operational formalas, that is, to the replacement of the operational equation

$$(4) \qquad \left(\sum_{s=0}^{\infty} c_s\, d^s\right) F = f$$

where F is a known function from which f is to be determined, and d is a finite dis-

Received November 1959; revised June 1960.

placement operator, by the equation

$$\left(\frac{U_{r,r}(d)}{V_{r,r}(d)}\right) F = f. \tag{5}$$

Equation (5) cannot at the moment, in its non-linear form, be solved. The equation may however be linearized by multiplication throughout by $V_{r,r}(d)$ to give

$$U_{r,r}(d)F = V_{r,r}(d)f. \tag{6}$$

Assuming that F and f are completely known, that r in equation (6) is sufficiently large, and the example is a suitable one, then there will exist considerable numerical agreement between the right and left hand sides of equation (6). Assuming that d is any one of the conventional operators Δ, E, ∇, μ, δ of numerical analysis, and that F and the sequences of values $f_1, f_2, \cdots; f_{-1}, f_{-2}, \cdots$ are known, then equation (6) may be rearranged so as to determine f_0. It is this very last assumption which constitutes a serious limitation of the linearizing technique resulting in equation (6). Indeed, Kopal was only able to find useful application of the technique when d was the backward difference operator, though his numerical results, which related to the forward integration of a differential equation, appeared to be very promising. However, the same effect over a very much larger range of problems may be achieved by recourse to another method.

3. The $e_m(S_n)$ Transformation. In his researches into the non-linear transformation*

$$e_m(S_n) = \frac{\begin{vmatrix} S_n & S_{n+1} & \cdots & S_{n+m} \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+m} \\ \vdots & \vdots & \cdots & \vdots \\ \Delta S_{n+m-1} & \Delta S_{n+m} & \cdots & \Delta S_{n+2m-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+m} \\ \vdots & \vdots & \cdots & \vdots \\ \Delta S_{n+m-1} & \Delta S_{n+m} & \cdots & \Delta S_{n+2m-1} \end{vmatrix}} \qquad m, n = 0, 1, \cdots \tag{7}$$

of the sequence S_r, $r = 0, 1, \cdots$ Shanks [5], by an appeal to the theory of linear equations, showed that if

$$S_r = \sum_{s=0}^{r} c_s x^s \qquad r = 0, 1, \cdots \tag{8}$$

then

$$e_m(S_n) = \frac{U_{m,m+n}(x)}{V_{m,m+n}(x)} \qquad m, n = 0, 1, \cdots. \tag{9}$$

The same result may be derived from the theory of orthogonal polynomials [6].

4. The ϵ-Algorithm. The evaluation of the determinants in the various expressions (7) is sufficiently laborious to be prohibitive. However, the expressions (7)

* The notation used here is consistent with that of [7] but differs slightly from that of [5] where the right hand side of (7) would be designated as $e_m(S_{n+m})$.

may be computed recursively by means of the ϵ-Algorithm as follows [7]. If, from the initial conditions

$$(10)\qquad \epsilon_{-1}^{(m)} = 0 \qquad m = 1, 2, \cdots ; \qquad \epsilon_0^{(m)} = S_m \qquad m = 0, 1, \cdots$$

quantities $\epsilon_s^{(m)}$ are computed recursively using the relation

$$(11)\qquad \epsilon_{s+1}^{(m)} = \epsilon_{s-1}^{(m+1)} + \frac{1}{\epsilon_s^{(m+1)} - \epsilon_s^{(m)}} \qquad m, s = 0, 1, \cdots,$$

then

$$(12)\qquad \epsilon_{2s+1}^{(m)} = \{e_s(\Delta S_m)\}^{-1} \quad \epsilon_{2s}^{(m)} = e_s(S_m) \qquad m, s = 0, 1, \cdots.$$

If the quantities $\epsilon_s^{(m)}$ are arranged in the scheme

$$\begin{array}{ccccccc}
 & \epsilon_0^{(0)} & & & & & \\
\epsilon_{-1}^{(1)} & & \epsilon_1^{(0)} & & & & \\
 & \epsilon_0^{(1)} & & \cdot & & & \\
\epsilon_{-1}^{(2)} & & \epsilon_1^{(1)} & & \cdot & \epsilon_s^{(0)} & \\
 & \epsilon_0^{(2)} & & \cdot & & & \epsilon_{s+1}^{(0)} \\
\epsilon_{-1}^{(3)} & & \epsilon_1^{(2)} & & \cdot & \epsilon_s^{(1)} & \cdot \\
 & \epsilon_0^{(3)} & & \cdot & \cdot & & \epsilon_{s+1}^{(1)} \\
\vdots & \vdots & \vdots & & \cdot & \epsilon_s^{(2)} & \cdot \\
 & & & & & & \epsilon_{s+1}^{(2)} \\
 & & & & & \vdots & \vdots \quad \cdot
\end{array}$$

it will be seen that relations (11) may be used, column by column, to build up the scheme from left to right. It should be noted that if conformity, by means of equations (9) and (12), is to take place between the Padé Table and the ϵ-array, the latter must be transposed about the diagonal $m = 0$; the columns of the ϵ-array with even order suffixes then take their place as rows in the Padé Table.

The following theorem, based upon the results of the last two sections, may now be given:

THEOREM. *If p is an associative and commutative operator, and*

$$(13)\qquad a_s p^s F = c_s x^s \qquad s = 0, 1, \cdots$$

and quantities $\epsilon_s^{(m)}$ are computed using the relation (11) *from the initial values*

$$(14)\qquad \epsilon_{-1}^{(m)} = 0 \qquad m = 1, 2, \cdots ; \qquad \epsilon_0^{(m)} = \sum_{s=0}^{m} a_s p^s F$$

then

$$(15)\qquad \epsilon_{2s}^{(m)} = \frac{U_{s,m+s}(x)}{V_{s,m+s}(x)} \qquad m, s = 0, 1, \cdots.$$

FIRST EXAMPLE: A numerical example of the application of the theorem now follows. It concerns the process of obtaining the derivative at $z = 0$ of the function $\exp(hz)$, when $h = 0.6$, by means of the formula

$$(16)\qquad \left(\frac{d}{dz} F\right)_{z=0} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s+1} \Delta^{s+1} F .$$

Here, in the notation of equation (13)

$$\frac{(-1)^s}{(s+1)}\Delta^{s+1}F = \frac{(-1)^s}{(s+1)}(e^h - 1)^{s+1}. \tag{17}$$

The quantities with even order suffix in the ϵ-array for this example are displayed in Table I.

Note: The results of Table I begin with the diagonal $m = 1$. If the notation of equation (15) is strictly to be adhered to, an entry $\epsilon_0^{(0)} = 0$ together with a corresponding diagonal should be appended to Table I. However, this is not a matter of great importance, and in the event that an operational series were to begin with a term in p^s, $s > 1$, even this artifice would not be available.

It is perhaps in order to comment upon the power of the algorithm as revealed by this example. Attainment of the same accuracy as is achieved in Table I by the straightforward use of the series (16), even neglecting the accumulation of round-off errors, would involve the summation of about eighty terms and an excursion into arithmetic involving twenty-eight decimal figures.

The Padé quotients (3) in this example are successive convergents of the continued fraction

$$x^{-1}\log(1+x) = \frac{1}{1+}\,\frac{1^2x}{2+}\,\frac{1^2x}{3+}\,\frac{2^2x}{4+}\,\frac{2^2x}{5+}\cdots. \tag{18}$$

Numerical investigation into the behavior of this continued fraction [8] shows that application of the ϵ-algorithm to the series (16) converges quite reasonably for $(e^{hs} - 1) > 1$, when the series rapidly diverges.

In the derivation of the classical operational formulas of numerical analysis the operand is assumed to be a polynomial, and the formulas derived are then completely valid. The formulas are then universally applied, without examination of the operand, and without any more justification than that of the results achieved.

In the same way it occurs that although formula (13) is no longer valid, use of the ϵ-algorithm in conjunction with operational series meets with success. Two examples which support this thesis now follow.

Second Example: This concerns the interpolation of the function $\log(0.6 + hz)$ when $h = 0.1$ and $z = 0.25$ with points of tabulation at unit intervals of z, by use

Table I

0.8221 1880					
0.4841 7914	0.6038 2270				
0.6693 9684	0.5987 5229	0.6000 7869			
0.5551 9362	0.6005 0406	0.5999 7937	0.6000 0168		
0.6303 0451	0.5997 6720	0.6000 0665	0.5999 9961	0.6000 0004	
0.5788 4612	0.6001 1785	0.5999 9752	0.6000 0011	0.5999 9999	0.6000 0000
0.6151 0747	0.5999 3622	0.6000 0102	0.5999 9996	0.6000 0000	
0.5890 2272	0.6000 3631	0.5999 9954	0.6000 0001		
0.6080 8473	0.5999 7849	0.6000 0022			
0.5939 8062	0.6000 1316				
0.6045 2176					

of Bessel's Interpolation Formula

$$F(z) = \sum_{s=0}^{\infty} a_s F(0) \tag{19}$$

where

$$\begin{aligned} a_0 &= 1 \qquad a_1 = z\delta E^{1/2} \\ a_{2s} &= \binom{2+s-1}{2s} \mu\delta^{2s} E^{1/2}, \\ a_{2s+1} &= \frac{z-\frac{1}{2}}{2s+1}\binom{2+s-1}{2s} \delta^{2s+1} E^{1/2} s = 1,2,\cdots. \end{aligned} \tag{20}$$

The quantities $\epsilon_s^{(m)}$ with even order suffix are displayed in Table II.

Since $\log(0.625) = -0.4700\ 036$, it will be seen that application of the ϵ-algorithm results in an effective gain of three decimal figures. This is not spectacular, but there is no point in selecting for presentation only those examples which display the method in a particularly favorable light. It might be mentioned at this point that the author has experimented with the ϵ-algorithm in conjunction with operational formulas in a large number of cases, and in none of these was the accuracy of the transformed results worse than the original partial sums.

Since the odd and even order terms in the series (19) are so dissimilar, the odd and even terms were separated out and the two series submitted separately to treatment by the ϵ-algorithm, the transformed results subsequently being added together. The numerical results produced in this way were not, however, significantly better than those shown in Table II.

Third Example: This concerns the application of the Euler-Maclaurin integration formula

$$\int_x^{x+h} F(t) = dt = \frac{h}{2}\{F(x) + F(x+h)\} - \sum_{s=1}^{\infty} \frac{h^{2s+1}B_{2s}}{(2s)!} \Delta_h F^{(2s-1)}(x) \tag{21}$$

when the integrand is the function $\exp(-z^2)$ and the upper and lower limits of integration are 0 and $w(1+i)$ respectively, with $w = 0.75$.

The functions $u_s = \dfrac{h^{s+2}F^{(s)}(h)}{(s+1)!}$ in this example satisfy the recursion

$$s(s+1)u_s + 2h^2 s u_{s-1} + 2h^2(s-1)u_{s-2} = 0 \tag{22}$$

Table II

−0.4337 503				
0.4722 880	−0.4701 290			
0.4700 009	0.4699 404	−0.4699 923		
0.4699 419	0.4699 732	0.4700 085	−0.4700 040	
0.4700 084	0.4700 105	0.4700 032	0.4700 027	−0.4700 034
0.4700 105	0.4700 088	0.4700 027	−0.4700 031	
0.4700 020	0.4700 017	0.4700 040		
0.4700 017	−0.4700 019			
−0.4700 048				

TABLE III

0.875 042 + *i*0.198 341					
0.941 289 0.385 856	0.921 529 + *i*0.408 848				
0.925 783 0.409 131	0.915 669 0.404 484	0.917 213 + *i*0.403 098			
0.915 984 0.408 382	0.917 028 0.402 979	0.917 498 0.403 703	0.917 256 + *i*0.403 708		
0.914 273 0.403 627	0.917 657 0.403 511	0.917 258 0.403 730	0.917 299 0.403 626	0.917 318 + *i*0.403 659	
0.916 792 0.401 460	0.917 403 0.403 874	0.917 271 0.403 620	0.917 321 0.403 657	0.917 302 0.403 657	0.917 307 + *i*0.403 652
0.919 060 0.402 800	0.917 147 0.403 734	0.917 332 0.403 635	0.917 302 0.403 661	0.917 306 + *i*0.403 651	
0.918 494 0.405 171	0.917 231 0.408 522	0.917 319 0.403 675	0.917 303 + *i*0.403 650		
0.915 922 0.405 283	0.917 427 0.403 573	0.917 289 + *i*0.403 663			
0.915 008 0.402 370	0.917 403 + *i*0.403 776				
0.918 429 + *i*0.400 273					

with

$$
(23) \qquad \begin{aligned} u_0 &= -w\{\cos(2w^2) + \sin(2w^2)\} - iw\{\cos(2w^2) - \sin(2w^2)\} \\ u_1 &= -2w^3\{\cos(2w^2) - \sin(2w^2)\} + i2w^3\{\cos(2w^2) + \sin(2w^2)\}. \end{aligned}
$$

The quantities $\epsilon_s^{(m)}$ (which are now complex numbers) with even order suffix are displayed in Table III. Since $\operatorname{erf}(0.75(1 + i)) = 0.917\ 306 + i0.403\ 654$, application of the ϵ-algorithm has in this case resulted in the gain of about three decimal places.

It is perhaps of interest to point out that the accuracy of the transformed results produced in the first and third examples could have been increased by extending the computation. This is also true to a limited extent of the second example, but the non-existence of central differences above a certain order limits the extent to which the computation may be prolonged.

It would be useful, when examining the mathematical validity of the procedures adopted in the second and third examples, to be able to relate the determinantal quotient

$$
\frac{\begin{vmatrix} \sum_{s=0}^{s=0} c_s p^s F & \sum_{s=0}^{s=1} c_s p^s F & \cdots & \sum_{s=0}^{s=m} c_s p^s F \\ c_1\, pF & c_2\, p^2 F & \cdots & c_{m+1}\, p^{m+1} F \\ \vdots & \vdots & & \vdots \\ c_m\, p^m F & c_{m+1}\, p^{m+1} F & \cdots & c_{2m}\, p^{2m} F \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ c_1\, pF & c_2\, p^2 F & \cdots & c_{m+1}\, p^{m+1} F \\ \vdots & \vdots & & \vdots \\ c_m\, p^m F & c_{m+1}\, p^{m+1} F & \cdots & c_{2m}\, p^{2m} F \end{vmatrix}}
$$

to the solution f of the operational equation

$$
\sum_{s=0}^{\infty} c_s p^s F = f,
$$

but this appears to be one of the cases in which a statement of the problem is not a great step forward to its solution.

5. **Acknowledgments.** This paper was written when the author was a member of the Istitut für Angewandte Mathematik in the University of Mainz; he is grateful to the Deutsche Forschungsgemeinschaft for a research grant which enabled the work to be carried out. The numerical results given were produced on the Z–22 at at Mainz.

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Expansion of Spherical Bessel Functions in a Series of Chebyshev Polynomials

By A. M. Arthurs and R. McCarroll

1. Introduction. In many problems of physics it is necessary to evaluate the spherical Bessel functions over a wide range of argument and up to a high order. For example, their evaluation is necessary in the solution of the integral equations of atomic scattering, as described in the work of Frazer [1].

One method is to generate the functions by means of recurrence formulas [2], [3], [4], which basically provide a large number of orders at a single value of the argument. A method which in essence provides a single order for a given *range* of argument, and which is suitable for use in automatic computations associated with atomic scattering, is to expand the spherical Bessel functions in terms of Chebyshev polynomials.

2. The Method. We introduce the shifted Chebyshev polynomial $T_n^*(z)$ which satisfies the following differential equation given by Lanczos [5].

$$(1) \qquad (z - z^2)\frac{d^2T_n^*}{dz^2} - \frac{(2z-1)}{2}\frac{dT_n^*}{dz} + n^2T_n^* = 0 \qquad 0 \leqq z \leqq 1.$$

The spherical Bessel function

$$(2) \qquad j_r(x) = (\pi/2x)^{1/2} J_{r+1/2}(x)$$

is expanded in series of $T_n^*(z)$ for $z \leqq 1$ as

$$(3) \qquad j_r(x) = (x/2)^r N_r \sum_n A_n T_n^*(z)$$

where

$$(4) \qquad z = x^2/p$$

and where N_r is the normalization factor given by

$$(5) \qquad N_r = \pi^{1/2}[2\,\Gamma(r + 3/2) \sum_n (-1)^n A_n]^{-1},$$

chosen so that as $x \to 0$

$$(6) \qquad j_r(x) \to \pi^{1/2}(x/2)^r/2\,\Gamma(r + 3/2).$$

The parameter p is chosen according to the required x-range.

Substituting the expression (3) into the differential equation satisfied by $j_r(x)$, namely,

$$(7) \qquad \left[\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + 1 - \frac{r(r+1)}{x^2}\right] j_r(x) = 0,$$

Received June 1, 1960.

Table 1

Tables of Expansion Coefficients

$$j_\nu(x) = (x/2)^\nu N_\nu \sum_n A_n T_n^*(x^2/100) \qquad -10 \leqq x \leqq 10$$

n	j_0 A_n	j_1 A_n
0	1.0000 0000 0000	1.0000 0000 0000
1	−1.8370 2985 7106	−1.7335 5787 9572
2	1.6181 7789 5224	1.7037 9637 8855
3	−2.4956 2512 4954	−1.2117 2623 8820
4	1.6832 2166 9128	0.4966 0881 5456
5	−0.5891 1635 2032	−0.1261 1219 3614
6	0.1275 7849 3735	0.0216 6172 6076
7	−0.0189 6827 0697	−0.0026 8726 9369
8	0.0020 6883 1574	0.0002 5255 1305
9	−0.0001 7326 9614	−0.0000 1863 3794
10	0.0000 1152 2733	0.0000 0110 9455
11	−0.0000 0062 4256	−0.0000 0005 4481
12	0.0000 0002 8116	0.0000 0000 2246
13	−0.0000 0000 1070	−0.0000 0000 0079
14	0.0000 0000 0035	0.0000 0000 0002
15	−0.0000 0000 0001	

n	j_2 A_n	j_3 A_n
0	1.0000 0000 0000	1.0000 0000 0000
1	−1.8364 1601 5317	−1.7972 4765 8075
2	1.4239 0223 9665	1.1197 5560 6070
3	−0.7040 0124 0486	−0.4322 1712 7394
4	0.2158 5586 1952	0.1074 1449 1493
5	−0.0438 6700 5066	−0.0183 5466 7482
6	0.0063 0718 4207	0.0022 8121 8406
7	−0.0006 7512 0822	−0.0002 1543 2710
8	0.0000 5593 1115	0.0000 1599 1357
9	−0.0000 0369 5921	−0.0000 0095 8213
10	0.0000 0019 9498	0.0000 0004 7352
11	−0.0000 0000 8968	−0.0000 0000 1964
12	0.0000 0000 0341	0.0000 0000 0069
13	−0.0000 0000 0011	−0.0000 0000 0002

n	j_4 A_n	j_5 A_n
0	1.0000 0000 0000	1.0000 0000 0000
1	−1.6836 0574 1497	−1.5488 1806 4538
2	0.8722 0517 6005	0.6859 0242 4243
3	−0.2782 8514 3648	−0.1872 6092 6662
4	0.0585 4062 0216	0.0342 7629 1788
5	−0.0086 6953 0055	−0.0044 9108 8777
6	0.0009 5173 1760	0.0004 4233 6046
7	−0.0000 8057 5832	−0.0000 3398 2101
8	0.0000 0542 5189	0.0000 0209 5486
9	−0.0000 0029 7662	−0.0000 0010 6104
10	0.0000 0001 3573	0.0000 0000 4494
11	−0.0000 0000 0523	−0.0000 0000 0162
12	0.0000 0000 0017	0.0000 0000 0005

n	j_6 A_n	j_7 A_n
0	1.0000 0000 0000	1.0000 0000 0000
1	−1.4171 3242 2748	−1.2972 4767 3744
2	0.5479 5705 4665	0.4451 8041 2800
3	−0.1310 4234 0105	−0.0948 5564 9030
4	0.0212 7626 4218	0.0138 5756 5596
5	−0.0025 0394 3447	−0.0014 8187 0216
6	0.0002 2390 8476	0.0001 2144 5196
7	−0.0000 1575 8616	−0.0000 0789 1023
8	0.0000 0089 6941	0.0000 0041 7246
9	−0.0000 0004 2186	−0.0000 0001 8329
10	0.0000 0000 1669	0.0000 0000 0680
11	−0.0000 0000 0056	−0.0000 0000 0022
12	0.0000 0000 0002	0.0000 0000 0001

Tables of Expansion Coefficients—Continued

n	j_8 A_n	j_9 A_n
0	1.0000 0000 0000	1.0000 0000 0000
1	−1.1911 3494 9922	−1.0981 6702 5095
2	0.3675 0203 3005	0.3078 0618 7533
3	−0.0706 7640 7499	−0.0539 7859 5631
4	0.0093 9388 4497	0.0065 8545 8180
5	−0.0009 2119 2296	−0.0005 9665 6926
6	0.0000 6972 0100	0.0000 4196 8280
7	−0.0000 0420 9178	−0.0000 0236 6972
8	0.0000 0020 7889	0.0000 0010 9704
9	−0.0000 0000 8569	−0.0000 0000 4260
10	0.0000 0000 0300	0.0000 0000 0141
11	−0.0000 0000 0009	−0.0000 0000 0004

n	j_{10} A_n	j_{11} A_n
0	1.0000 0000 0000	1.0000 0000 0000
1	−1.0168 7201 6138	−0.9456 3737 0019
2	0.2611 5956 8765	0.2241 3285 9656
3	−0.0421 0983 2180	−0.0334 5827 7279
4	0.0047 5003 6264	0.0035 1079 6899
5	−0.0004 0010 7956	−0.0002 7638 9359
6	0.0000 2629 5870	0.0000 1705 0581
7	−0.0000 0139 1880	−0.0000 0085 0428
8	0.0000 0006 0782	0.0000 0003 5115
9	−0.0000 0000 2232	−0.0000 0000 1223
10	0.0000 0000 0070	0.0000 0000 0036
11	−0.0000 0000 0002	−0.0000 0000 0001

n	j_{12} A_n	j_{13} A_n
0	1.0000 0000 0000	1.0000 0000 0000
1	−0.8829 7010 1098	−0.8275 7427 4817
2	0.1943 1493 5452	0.1699 8535 0546
3	−0.0270 1121 0330	−0.0221 1284 8562
4	0.0026 5020 1752	0.0020 3772 5111
5	−0.0001 9588 4596	−0.0001 4196 6316
6	0.0000 1138 8223	0.0000 0780 5415
7	−0.0000 0053 7112	−0.0000 0034 9193
8	0.0000 0002 1036	0.0000 0001 3008
9	−0.0000 0000 0697	−0.0000 0000 0411
10	0.0000 0000 0020	0.0000 0000 0011

n	j_{14} A_n	j_{15} A_n
0	1.0000 0000 0000	1.0000 0000 0000
1	−0.7783 5629 2813	−0.7344 0438 9543
2	0.1498 9703 5766	0.1331 3186 9771
3	−0.0183 2686 5868	−0.0153 5579 5806
4	0.0015 9235 3331	0.0012 6227 9463
5	−0.0001 0493 0348	−0.0000 7891 7018
6	0.0000 0547 2793	0.0000 0391 5367
7	−0.0000 0023 2885	−0.0000 0015 8871
8	0.0000 0000 8272	0.0000 0000 5393
9	−0.0000 0000 0250	−0.0000 0000 0156
10	0.0000 0000 0006	0.0000 0000 0004

TABLE 2
Table of Normalization Factors

r	N_r
0	$10^{-1}\cdot 1.0670\ 1130\ 3957$
1	$10^{-1}\cdot 1.0588\ 0224\ 7273$
2	$10^{-2}\cdot 5.0977\ 3196\ 1602$
3	$10^{-2}\cdot 1.7016\ 2862\ 5983$
4	$10^{-3}\cdot 4.3387\ 2970\ 2098$
5	$10^{-4}\cdot 8.8939\ 6388\ 4585$
6	$10^{-4}\cdot 1.5178\ 7598\ 0079$
7	$10^{-5}\cdot 2.2135\ 3657\ 5215$
8	$10^{-6}\cdot 2.8143\ 4241\ 8551$
9	$10^{-7}\cdot 3.1696\ 2427\ 4560$
10	$10^{-8}\cdot 3.2028\ 4879\ 3012$
11	$10^{-9}\cdot 2.9343\ 5222\ 5384$
12	$10^{-10}\cdot 2.4587\ 5268\ 1163$
13	$10^{-11}\cdot 1.8981\ 3134\ 3689$
14	$10^{-12}\cdot 1.3584\ 9275\ 4731$
15	$10^{-14}\cdot 9.0623\ 7664\ 9006$

and using the properties of the shifted Chebyshev polynomials, we obtain a set of simultaneous linear algebraic equations which may be solved for the ratios A_1/A_0, A_2/A_0, $\cdots$. The solutions may be normalized, using the factors given by (5).

3. Calculations and Results. The calculation of the coefficients A_n has been carried out on a DEUCE digital computer for $p = 100$, and $r = 0$ to 15. For all values of r, A_0 has been chosen as 1.0, and the ratios A_n/A_0 computed accordingly. Tables of the coefficients, together with the corresponding normalization factors are presented herein. The expansion coefficients are given to 12 decimal places to insure that for the range of x considered the spherical Bessel functions should be accurate to 10 significant figures.

As can be seen from the tables, the convergence of the coefficients is very rapid; if the Chebyshev expansion and Taylor series are curtailed after n terms, the ratio of the $(n + 1)$th terms is about $1/2^{n-1}$.

The authors wish to thank Mrs. N. Scott and Miss M. Wilson for assistance with the computations.

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Gaussian Quadrature with Weight Function x^n on the Interval $(-1, 1)$

By Harry A. Rothmann

Most of the existent literature dealing with Gaussian quadrature formulas is based on the assumption that the relevant weight function is of constant sign in the interval of integration. This note deals with the weight function x^n on the interval $(-1, 1)$ and indicates the extent to which the existent theory can be generalized in that case.

Gaussian quadrature formulas on the interval $(-1, 1)$ have the form

$$\int_{-1}^{1} w(x)f(x)\,dx = \sum_{k=1}^{m} W_k f(x_k) + E \tag{1}$$

where the weights and abscissas are chosen to ensure a degree of precision $2m - 1$ (i.e., exactness for all polynomials of degree not exceeding $2m - 1$).

One method of determining the abscissas x_k involves obtaining a set of polynomials, $\phi_1, \phi_2, \phi_3, \cdots$, such that each is orthogonal to all polynomials of inferior degree relative to the weight function $w(x)$ over the interval $(-1, 1)$. That is, for the mth-degree polynomial $\phi_m(x)$,

$$\int_{-1}^{1} w(x)\phi_m(x)q_{m-1}(x)\,dx = 0 \tag{2}$$

where q_{m-1} is an arbitrary polynomial of degree $m - 1$ or less. The m abscissas x_k then are the zeros of the polynomial ϕ_m.

Upon defining

$$w(x)\phi_m(x) = \frac{d^m U_m(x)}{dx^m} \tag{3}$$

the requirement that ϕ_m be of degree m implies that $U_m(x)$ must satisfy the differential equation

$$\frac{d^{m+1}}{dx^{m+1}}\left[\frac{1}{w(x)}\frac{d^m U_m(x)}{dx^m}\right] = 0 \tag{4}$$

in the interval $(-1, 1)$. From expression (2), the $2m$ boundary conditions

$$U_m(\pm 1) = U_m'(\pm 1) = \cdots = U_m^{(m-1)}(\pm 1) = 0 \tag{5}$$

can be obtained. When $w(x) = x^n$, the solution of (4) is found to be of the form

$$U_m(x) = x^m w(x)[c_0 + c_1 x + \cdots + c_m x^m] + d_0 + d_1 x + \cdots + d_{m-1}x^{m-1}$$

from which ϕ_m can be obtained. It is convenient to impose the additional normalizing property

$$\phi_m(1) = 1. \tag{6}$$

Received June 6, 1960.

From the theory as given in [1], the weight corresponding to the abscissas x_k for the N-point formula is

$$W_{N,k} = \frac{1}{\phi_N'(x_k)} \int_{-1}^{1} w(x) \frac{\phi_N(x)\, dx}{x - x_k}. \tag{7}$$

When the weight function $w(x)$ is of constant sign, (7) reduces to

$$W_{N,k} = \frac{A_N \gamma_{N-1}}{A_{N-1}\phi_N'(x_k)\phi_{N-1}(x_k)} = -\frac{A_{N+1}\gamma_N}{A_N \phi_N'(x_k)\phi_{N+1}(x_k)} \tag{8}$$

where A_N is the coefficient of x^N in ϕ_N and

$$\gamma_N = A_N \int_{-1}^{1} w(x) x^N \phi_N(x)\, dx. \tag{9}$$

The error function for the N-point formula can be expressed as

$$E_N = \int_{-1}^{1} w(x) f[x_1, x_1, \cdots, x_N, x_N, x]\pi_N^2(x)\, dx \tag{10}$$

with $\pi_N(x) = (x - x_1)(x - x_2) \cdots (x - x_N)$ where the x_i are the zeros of ϕ_N. The expression (10) reduces in the case when $w(x)$ is of constant sign to

$$E_N = \frac{\gamma_N f^{(2N)}(\eta)}{A_N^2(2N)!} \tag{11}$$

where $f(x)$ is assumed to have $2N$ continuous derivatives in $(-1, 1)$ and η is some value in that interval.

When $w(x) = x^{2n}$, the procedure outlined above can be employed to obtain the following

$\Phi_0 = 1 \qquad \Phi_1(x) = x$

$$\Phi_2(x) = \frac{1}{2}[(2n+3)x^2 - (2n+1)]$$

$$\Phi_3(x) = \frac{1}{2}[(2n+5)x^3 - (2n+3)x]$$

$$\Phi_4(x) = \frac{1}{2!\,2^2}[(2n+7)(2n+5)x^4 - 2(2n+5)(2n+3)x^2 + (2n+3)(2n+1)]$$

$$\Phi_5(x) = \frac{1}{2!\,2^2}[(2n+9)(2n+7)x^5 - 2(2n+7)(2n+5)x^3 + (2n+5)(2n+3)x].$$

From these, the forms

$$\Phi_{2N}(x) = \frac{1}{N!\,2^N}\sum_{i=0}^{N}\binom{N}{i}(-1)^i \prod_{j=1}^{N}[2n + 2(N-i) + 2j - 1]\,x^{2(N-i)} \tag{12}$$

$$(m = 2N)$$

$$\Phi_{2N+1}(x) = \frac{1}{N!\,2^N}\sum_{i=0}^{N}\binom{N}{i}(-1)^i \prod_{j=1}^{N}[2n + 2(N-i) + 2j + 1]\,x^{2(N-i)+1} \tag{13}$$

$$(m = 2N + 1)$$

can be deduced [2] by an inductive procedure and shown to satisfy the desired properties (2) and (6). Since the weight function is of constant sign in the interval $(-1, 1)$, a known theorem [1, page 171–2] states that the zeros of the polynomials are all real, distinct, and lie in the interval $(-1, 1)$.

To display the formulas for the weights and error, the values A_N and γ_N must be obtained. From (13) and (13), it is easily seen that

$$(14)\qquad \begin{aligned} A_{2N} &= \frac{1}{N!\,2^N}\prod_{j=1}^{N}(2n+2N+2j-1) \\ A_{2N+1} &= \frac{1}{N!\,2^N}\prod_{j=1}^{N}(2n+2N+2j+1). \end{aligned}$$

The value

$$(15)\qquad \gamma_N = \frac{2}{2n+2N+1}$$

can be established by an inductive proof [2]. Thus, the weights are now expressed in the form

$$\begin{aligned} W_{2N,k} &= \frac{1}{N\Phi'_{2N}(x_k)\Phi_{2N-1}(x_k)} = \frac{-2}{(2n+2N+1)\Phi'_{2N}(x_k)\Phi_{2n+1}(x_k)} \\ W_{2N+1,k} &= \frac{2}{(2n+2N+1)\Phi'_{2N+1}(x_k)\Phi_{2N}(x_k)} = \frac{-1}{(N+1)\Phi_{2N+1}(x_k)\Phi_{2N+2}({}_k)}. \end{aligned}$$

Since $w(x)$ is of constant sign, the error is given by (11) which can be written in the form

$$\begin{aligned} E_{2N} &= \frac{2(N!\,2^N)^2 f^{(4N)}(\eta)}{(4N)!\,(2n+4N+1)\prod_{j=1}^{N}(2n+2N+2j-1)^2} \\ E_{2N+1} &= \frac{2(N!\,2^N)^2 f^{(4N+2)}(\eta)}{(4N+2)!\,(2n+4N+3)\prod_{j=1}^{N}(2n+2N+2j+1)^2} \qquad (-1<\eta<1). \end{aligned}$$

When $w(x) = x^{2n+1}$, it is found [2] that the formulas for an odd number of abscissas do not exist, but that the polynomials of even degree satisfying properties (2) and (6) do exist, and have the form

$$(16)\qquad \theta_{2N}(x) = \frac{1}{N!\,2^N}\sum_{i=0}^{N}\binom{N}{i}(-1)^i\prod_{j=1}^{N}[2n+2(N-i)+2j+1]\,x^{2(N-i)}.$$

Since $x\theta_{2N}(x) = \Phi_{2N+1}(x)$, the zeros of θ_{2N} are all real and distinct and lie in the interval $(-1, 1)$, and, in fact, are the zeros of Φ_{2N+1} when the zero $x = 0$ is suppressed.

The weights for the $2N$-point formula involve the polynomial Φ_{2N+1} in the following manner

$$W_{2N,k} = \frac{1}{\theta'_{2N}(x_k)}\int_{-1}^{1} x^{2n+1}\frac{\theta_{2N}(x)}{x-x_k}\,dx$$

$$= \frac{1}{\theta'_{2N}(x_k)} \int_{-1}^{1} x^{2n} \frac{\Phi_{2N+1}(x)}{x - x_k}\, dx$$

which reduces to

$$(17) \qquad W_{2N,k} = \frac{-A_{2N+2}\,\gamma_{2N+1}}{A_{2N+1}\theta'_{2N}(x_k)\Phi_{2N+2}(x_k)} = \frac{A_{2N+1}\,\gamma_{2N}}{A_{2N}\theta'_{2N}(x_k)\Phi_{2N}(x_k)}$$

where the A's and γ's refer to the polynomial Φ and are given by (14) and (15). After the substitution of (14) and (15), (17) reduces to

$$W_{2N,k} = \frac{-1}{(N+1)\theta'_{2N}(x_k)\Phi_{2N+2}(x_k)} = \frac{2}{(2n+2N+1)\theta'_{2N}(x_k)\Phi_{2N}(x_k)}.$$

From (10), the error term E_{2N} can be written as

$$E_{2N} = \int_{-1}^{1} x^{2n+1} f[x_1, x_1, \cdots, x_{2N}, x_{2N}, x]\, \pi_{2N}^2(x)\, dx$$

TABLE 1

Gaussian Quadrature with Weight x^{2n}

m	n	Weights	Abscissas	Error Coefficients
2	0	1.0000000	± .5773503	7.4×10^{-3}
	1	.3333333	± .7745967	1.9×10^{-3}
	2	.2000000	± .8451543	7.6×10^{-4}
	3	.1428571	± .8819171	3.7×10^{-4}
	4	.1111111	± .9045340	2.1×10^{-4}
	5	.09090909	± .9198662	1.3×10^{-4}
3	0	.8888889	0	6.3×10^{-5}
		.5555556	± .7745967	
	1	.1066667	0	2.5×10^{-5}
		.2800000	± .8451543	
	2	.03265306	0	1.2×10^{-5}
		.1836735	± .8819171	
	3	.01410935	0	7.1×10^{-6}
		.1358025	± .9045340	
	4	.007346189	0	4.4×10^{-6}
		.1074380	± .9198662	
	5	.004303389	0	2.9×10^{-6}
		.08875740	± .9309493	
4	0	.3478548	± .8611363	2.9×10^{-7}
		.6521452	± .3399810	
	1	.1945553	± .9061798	7.3×10^{-8}
		.1387780	± .5384693	
	2	.1343622	± .9290483	2.5×10^{-8}
		.06563784	± .6399973	
	3	.1024498	± .9429254	1.0×10^{-8}
		.04040730	± .7039226	
	4	.08273203	± .9522526	4.9×10^{-9}
		.02837808	± .7482524	
	5	.06935661	± .9589554	2.6×10^{-9}
		.02155248	± .7809074	

which reduces after one integration by parts to

$$E_{2N} = -\int_{-1}^{1} f[x_1, x_1, \cdots, x_{2N}, x_{2N}, x, x] \int_{-1}^{x} x^{2n+1} \pi_{2N}^2(x)\, dx\, dx.$$

Now consider the function

$$A(x) = \int_{-1}^{x} x^{2n+1} \pi_{2N}^2(x)\, dx.$$

Since $A'(x)$ is negative for $x < 0$ and positive for $x > 0$ and $A(\pm 1) = 0$, it follows that $A(x)$ is of constant sign in $(-1, 1)$ and hence

$$\begin{aligned} E_{2N} &= -f[x_1, \cdots, x_{2N}, \xi, \xi] \int_{-1}^{1} \int_{-1}^{x} x^{2n+1} \pi_{2N}^2(x)\, dx && (-1 < \xi < 1) \\ &= \frac{f^{(4N+1)}(\eta)}{(4N+1)!} \int_{-1}^{1} x^{2n} K_{2N+1}^2(x)\, dx && (-1 < \eta < 1) \end{aligned} \tag{18}$$

where $f(x)$ is assumed to have $4N + 1$ continuous derivatives in $(-1, 1)$. The function K_{2N+1} is the monic polynomial consisting of the linear factors of Φ_{2N+1} and therefore (18) reduces to

$$E_{2N} = \frac{f^{(4N+1)}(\eta)\gamma_{2N+1}}{(4N+1)!\, A_{2N+1}^2}$$

i. e.

$$E_{2N} = \frac{2(N!\,2^N)^2 f^{(4N+1)}(\eta)}{(4N+1)!(2n+4N+3)\prod_{j=1}^{N}(2n+2N+2j+1)^2}.$$

TABLE 2

Gaussian Quadrature with Weight x^{2n+1}

m	n	Weights	Abscissas	Error coefficients
2	0	±.4303315	±.7745967	3.8×10^{-4}
	1	±.2366432	±.8451543	1.5×10^{-4}
	2	±.1619848	±.8819171	7.5×10^{-5}
	3	±.1228380	±.9045340	4.2×10^{-5}
	4	±.09882860	±.9198662	2.6×10^{-5}
	5	±.08262864	±.9309493	1.7×10^{-5}
4	0	±.2577268	±.5384693	8.1×10^{-9}
		±.2146984	±.9061798	
	1	±.1025596	±.6399973	2.8×10^{-9}
		±.1446234	±.9290483	
	2	±.05740305	±.7039226	1.1×10^{-9}
		±.1086511	±.9429254	
	3	±.03792714	±.7482524	5.5×10^{-10}
		±.08688035	±.9522526	
	4	±.02759928	±.7809074	2.9×10^{-10}
		±.07232517	±.9589554	
	5	±.02137648	±.8060023	1.6×10^{-10}
		±.06192242	±.9640060	

In Tables 1 and 2, the weights and abscissas are given for $m = 2, 3, 4$, for $w(x) = x^{2n}$ and for $m = 2, 4$, in the case $w(x) = x^{2n+1}$. The values for the weights and abscissas are correct to the 7 figures given. The coefficient of $f^{(s)}(\eta)$ in the error term is also given. These error coefficients are correct to the two figures given.

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A Table of Generalized Circular Error

By Harry Weingarten and A. R. Di Donato

1. Introduction. This note provides an abbreviated table (Table 1) giving solutions for the value of K satisfying

$$\frac{1}{2\pi\sigma_x\sigma_y}\int_R\int \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)\right]dx\,dy = P$$

where R is the circle $x^2 + y^2 = K^2\sigma_x^2$, $\sigma_x \geqq \sigma_y$ and $c = \sigma_y/\sigma_x$. K has been computed for $c = 0(.01)1$ and $P = 0(.01).99$. The table provided here will not contain all the results because of space limitations. The complete table is available upon request directed to either author. It differs from the extensive one in [1] which also gives numerous applications and a wide bibliography of the bivariate normal distribution.

2. Application. When $P = .5$ and $c = 1$ we obtain the cPE (circular probable error) relationship used in ballistic studies. In this case $K = 1.17741$, which may easily be found without the table in this note. When $c \neq 1$, however, (which is the usual case) it is still of interest to find the circles within which impacts will occur with given probabilities, rather than the ellipses. For any particular P and c the value of K in the table multiplied by σ_x is the required radius.

3. Statistical Interpretation. This kind of problem has been widely considered as indicated by the references in [2], where the approach is differently oriented, being concerned with the general problem of the distribution of quadratic forms. Essentially we consider here the cumulative distribution in tabular form of the random variable,

$$Z = X^2 + Y^2$$

where X and Y are independently and normally distributed with zero means and variances σ_x and σ_y. (Z does not, of course, have a χ^2 distribution unless $\sigma_x = \sigma_y = 1$.) In [3], Chapter 27, there will be found application of such results to the specification of regions of type C in the testing of hypotheses.

4. Analysis. This section will detail the computational and numerical analysis aspects of the preparation of the table.

The probability integral under consideration is given by the following equation:

$$(1)\qquad P(K, \sigma_x, \sigma_y) = \frac{1}{2\pi\sigma_x\sigma_y}\int_R\int \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)\right]dx\,dy$$

where the region, R, is specified as a circle with its center at the origin and with radius $K\sigma_x$. The use of polar coordinates transforms (1) to

$$(2)\qquad P(K, c) = \frac{1}{2\pi c}\int_0^{2\pi}\int_0^{K} \exp\left[-\tfrac{1}{2}\rho^2\left(\frac{1+c^2}{2c^2}-\frac{1-c^2}{2c^2}\cos 2\theta\right)\right]\rho\,d\rho\,d\theta$$

Received June 27, 1960.

Table 1

The Generalized Circular Probable Error K

P \ c	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
.05	0.08149	0.10697	0.12806	0.14627	0.16251	0.17730	0.19097	0.20375	0.21579	0.22721
.10	0.13631	0.16328	0.19017	0.21449	0.23662	0.25701	0.27599	0.29383	0.31070	0.32675
.15	0.19590	0.21757	0.24565	0.27316	0.29897	0.32313	0.34585	0.36734	0.38777	0.40729
.20	0.25834	0.27454	0.30048	0.32894	0.35690	0.38367	0.40917	0.43349	0.45676	0.47910
.25	0.32259	0.33506	0.35715	0.38470	0.41348	0.44188	0.46941	0.49596	0.52155	0.54624
.30	0.38858	0.39867	0.41685	0.44210	0.47050	0.49965	0.52853	0.55677	0.58424	0.61093
.35	0.45653	0.46500	0.48004	0.50225	0.52924	0.55829	0.58788	0.61731	0.64626	0.67463
.40	0.52679	0.53409	0.54679	0.56592	0.59073	0.61889	0.64854	0.67866	0.70872	0.73846
.45	0.59986	0.60623	0.61721	0.63363	0.65585	0.68244	0.71154	0.74184	0.77260	0.80339
.50	0.67635	0.68199	0.69163	0.70585	0.72543	0.74994	0.77788	0.80785	0.83890	0.87042
.55	0.75707	0.76210	0.77066	0.78314	0.80039	0.82243	0.84870	0.87782	0.90870	0.94060
.60	0.84311	0.84761	0.85527	0.86634	0.88142	0.90113	0.92532	0.95307	0.98332	1.01520
.65	0.93593	0.93998	0.94685	0.95675	0.97008	0.98751	1.00939	1.03532	1.06444	1.09586
.70	1.03764	1.04129	1.04748	1.05635	1.06822	1.08361	1.10311	1.12685	1.15433	1.18481
.75	1.15144	1.15473	1.16029	1.16825	1.17884	1.19246	1.20968	1.23100	1.25637	1.28534
.80	1.28253	1.28548	1.29046	1.29759	1.30704	1.31908	1.33421	1.35302	1.37588	1.40275
.85	1.44040	1.44303	1.44746	1.45379	1.46215	1.47277	1.48599	1.50233	1.52238	1.54653
.90	1.64561	1.64791	1.65179	1.65731	1.66461	1.67383	1.68523	1.69918	1.71626	1.73708
.95	1.96060	1.96253	1.96578	1.97041	1.97651	1.98420	1.99366	2.00514	2.01902	2.03586
.96	2.05436	2.05620	2.05930	2.06371	2.06953	2.07686	2.08587	2.09679	2.10995	2.12588
.97	2.17067	2.17241	2.17534	2.17952	2.18502	2.19195	2.20045	2.21075	2.22314	2.23806
.98	2.32689	2.32851	2.33124	2.33514	2.34026	2.34672	2.35464	2.36421	2.37569	2.38948
.99	2.57632	2.57778	2.58025	2.58377	2.58839	2.59421	2.60134	2.60995	2.62025	2.63257

P \ c	.55	.60	.65	.70	.75	.80	.85	.90	.95	1.0
.05	0.23810	0.24852	0.25854	0.26820	0.27753	0.28657	0.29534	0.30388	0.31219	0.32029
.10	0.34210	0.35683	0.37101	0.38472	0.39798	0.41085	0.42336	0.43555	0.44744	0.45904
.15	0.42601	0.44402	0.46142	0.47825	0.49458	0.51045	0.52591	0.54099	0.55571	0.57012
.20	0.50060	0.52136	0.54145	0.56094	0.57990	0.59835	0.61636	0.63396	0.65118	0.66805
.25	0.57012	0.59326	0.61573	0.63758	0.65888	0.67967	0.69999	0.71989	0.73939	0.75853
.30	0.63688	0.66213	0.68672	0.71072	0.73418	0.75712	0.77960	0.80166	0.82331	0.84460
.35	0.70237	0.72950	0.75604	0.78202	0.80748	0.83246	0.85699	0.88110	0.90483	0.92821
.40	0.76775	0.79655	0.82486	0.85268	0.88004	0.90696	0.93346	0.95958	0.98534	1.01077
.45	0.83399	0.86428	0.89421	0.92375	0.95291	0.98170	1.01013	1.03822	1.06599	1.09347
.50	0.90207	0.93365	0.96505	0.99621	1.02709	1.05769	1.08801	1.11807	1.14786	1.17741
.55	0.97303	1.00569	1.03841	1.07107	1.10361	1.13599	1.16819	1.20021	1.23206	1.26373
.60	1.04810	1.08162	1.11549	1.14954	1.18366	1.21779	1.25187	1.28590	1.31985	1.35373
.65	1.12888	1.16298	1.19781	1.23312	1.26875	1.30460	1.34058	1.37666	1.41281	1.44901
.70	1.21752	1.25187	1.28742	1.32384	1.36090	1.39845	1.43637	1.47459	1.51306	1.55176
.75	1.31724	1.35143	1.38739	1.42471	1.46309	1.50231	1.54222	1.58271	1.62369	1.66511
.80	1.43320	1.46668	1.50262	1.54055	1.58010	1.62096	1.66294	1.70586	1.74962	1.79412
.85	1.57477	1.60677	1.64206	1.68015	1.72059	1.76302	1.80717	1.85280	1.89974	1.94788
.90	1.76212	1.79152	1.82511	1.86253	1.90335	1.94716	1.99359	2.04236	2.09321	2.14597
.95	2.05638	2.08130	2.11111	2.14598	2.18580	2.23029	2.27908	2.33180	2.38812	2.44775
.96	2.14527	2.16891	2.19748	2.23134	2.27054	2.31491	2.36413	2.41782	2.47565	2.53727
.97	2.25619	2.27835	2.30537	2.33788	2.37617	2.42021	2.46978	2.52455	2.58415	2.64823
.98	2.40614	2.42650	2.45153	2.48214	2.51895	2.56226	2.61202	2.66799	2.72983	2.79715
.99	2.64736	2.66533	2.68750	2.71505	2.74916	2.79069	2.84010	2.89743	2.96249	3.03485

where ρ, θ are the usual polar coordinates stretched by a factor σ_x and where

$$0 \leq \frac{\sigma_y}{\sigma_x} = c \leq 1. \tag{3}$$

Simple transformations reduce $P(K, c)$ to

$$P(K, c) = \frac{1}{\pi c} \int_0^{K^2/2} e^{-Bw} \int_0^{\pi} e^{Aw\cos\theta} \, d\theta \, dw \tag{4}$$

where

$$A = \frac{1 - c^2}{2c^2} \quad \text{and} \quad B = \frac{1 + c^2}{2c^2}. \tag{5}$$

The integral over θ in (4) is referred to in [4, (p. 46)], and may be expressed as

$$\int_0^{\pi} e^{Aw\cos\theta} \, d\theta = \pi I_0(Aw) \tag{6}$$

where $I_0(x)$ is defined by the following Taylor and asymptotic expansions respectively

$$I_0(x) = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right)^2 \left(\frac{x}{2}\right)^{2n} \tag{7}$$

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}} \sum_{n=0}^{N'} \frac{[(2n)!]^2}{2^{5n}(n!)^3} x^{-n}. \tag{8}$$

The relations (7) and (8) are given in [4] on pages 20 and 58, respectively.

Two computation schemes were used for computing P. If $AK^2 \leq 40$ (an arbitrary choice), then the following recurrence relation was used to compute P:

$$T_{2n} = \frac{2n-1}{2n} \left(\frac{A}{B}\right)^2 T_{2n-2} - \frac{1}{Bc} \left[\left(\frac{AK^2}{4}\right)^{2n-1} e^{-BK^2/2} \left\{\frac{AK^2}{4} + n\left(\frac{A}{B}\right)\right\}\right] \left(\frac{1}{n!}\right)^2 \tag{9}$$

where

$$T_0 = \frac{1}{Bc}(1 - e^{-BK^2/2}) \tag{10}$$

and

$$P = \sum_{n=0}^{N} T_{2n}, \qquad AK^2 \leq 40. \tag{11}$$

If $AK^2 > 40$ then the following recurrence relation was used to compute P:

$$M_{2n+1} = \frac{1}{2Ac} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{2}{2n-1} \frac{[(2n)!]^2}{2^{4n}(n!)^3} (AK^2)^{-(2n-1)/2} e^{-K^2/2} - \frac{1}{2A} \frac{2n-1}{2n} M_{2n-1} \tag{12}$$

where

$$M_1 = \frac{1}{\sqrt{1-c^2}} \cdot \frac{2}{\sqrt{\pi}} \int_{K/\sqrt{2}}^{\infty} e^{-w^2} \, dw \tag{13}$$

and

$$P = 1 - \sum_{n=0}^{N'} M_{2n+1}. \tag{14}$$

Equation (9) is obtained by substituting (7) into (4), transforming the upper limit on the W integration from $K^2/2$ to $AK^2/4$, interchanging summation and integration, and then performing two successive integrations by parts on the integral that occurs as part of the general nth term of the series. The upper limit, N, of the sum that appears in (11) is determined when

$$| T_{2N} | \leqq \epsilon \left| \sum_{n=0}^{N} T_{2n} \right| \tag{15}$$

where ϵ is chosen to the order of accuracy to which P is desired.

The recurrence relation given by (12) is derived by substituting (8) into (4), interchanging summation and integration, and by considering the integral from AK^2 to infinity rather than from 0 to AK^2. Two integrations by parts on the integral that occurs as part of the general nth term of the series yield (12). The integer N' is determined such that

$$| M_{2N'+1} | \leqq \epsilon \left| \sum_{n=0}^{N'} M_{2n+1} \right|. \tag{16}$$

The restriction of (12) to the region $AK^2 > 40$ insures at least eight-digit accuracy in P before the terms M_{2n+1} eventually begin to increase in magnitude. The ϵ's in (15) and (16) were set at 10^{-8}.

Inasmuch as equal intervals in P and c were desired, a Newton-Raphson procedure was used to determine K for a given P and c; accordingly

$$K_n = K_{n-1} - \frac{\frac{1}{c} \int_0^{(K_{n-1}^2/2)} e^{-Bw} I_0(Aw)\, dw - P}{\frac{1}{c} K_{n-1} e^{-(BK_{n-1}^2/2)} I_0\left(\frac{AK_{n-1}^2}{2}\right)} \tag{17}$$

where K_n represents the nth iterate for K.

The efficiency and accuracy of the computation are indicated by the fact that the average time required to evaluate a K to eight significant digits for a given P and c was 50 milliseconds on NORC. The accuracy of the results was checked by evaluating the same K by both (9) and (12) in the region of $AK^2 = 40$. Thirty terms were used for this purpose. This region is where both series (11) and (14) require the largest number of terms, and consequently where truncation and rounding errors should be the largest. Some further checks to insure eight-digit accuracy were obtained by evaluating some of the integrals by the direct application of Simpson's Rule. The entire table presented herein required less than 30 seconds of computing time on NORC.

5. Acknowledgment. The problem itself was presented to the Naval Weapons Laboratory, Dahlgren, Virginia, by the first author named. The second author carried out the actual solution of the problem as given in section 4, above. Both authors

wish to thank Mr. Irv Learman, Naval Weapons Laboratory, who did the programming and coding, and Mr. John Walker, Naval Weapons Laboratory, who programmed and coded the editing procedure for setting up the table.

Special Projects Office
Navy Department
Washington, District of Columbia
Naval Weapons Laboratory
Dahlgren, Virginia

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Polynomial Approximations to Integral Transforms

By Jet Wimp

1. Introduction. The symmetric Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$, orthogonal on the interval $-1 \leqq x \leqq 1$, are widely used for approximating functions, but the integral which defines the coefficients for the expansion of a function $g(x)$ in these polynomials usually is quite difficult to evaluate. The problem is simplified if $g(x)$ is an integral transform of the Fourier or Laplace type, since the kernel of the transform generates a series of the above polynomials. The coefficients in such cases are found to be Hankel transforms, which are widely tabulated.

Examples include Chebyshev polynomial expansions of $1/(x + a)^k$, $\psi(x + a)$, $\log \Gamma(x + a)$, $Ci(x)$ and $Si(x)$.

2. Formulas When $g(x)$ is a Laplace or Fourier Transform. The symmetric Jacobi polynomials [1, v. 2, p. 168] may be defined by

$$P_n^{(\alpha,\alpha)}(x) = \binom{n+\alpha}{n} {}_2F_1[-n, n + 2\alpha + 1; \alpha + 1; \tfrac{1}{2} - \tfrac{1}{2}x]. \tag{1}$$

A function $g(x)$ satisfying certain conditions has the expansion

$$g(x) = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\alpha)}(x), \qquad -1 \leqq x \leqq 1, \tag{2}$$

where

$$A_n = \frac{(2n + 2\alpha + 1)n!\Gamma(n + 2\alpha + 1)}{2^{2\alpha+1}[\Gamma(n + \alpha + 1)]^2} \int_{-1}^{1} g(x)(1 - x^2)^\alpha P_n^{(\alpha,\alpha)}(x)\, dx. \tag{3}$$

Suppose now that $g(x)$ is the Laplace transform of some $f(t)$,

$$g(x) = \mathfrak{L}\{f(t)\} = \int_0^\infty e^{-xt} f(t)\, dt = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\alpha)}(x). \tag{4}$$

To determine the A_n's replace the kernel of the Laplace transform by its Neumann series [1, v. 2: p. 98, No. (1); p. 175, No. (16); p. 174, No. (6); and the duplication formula for the gamma function].

$$e^{-xt} = \sum_{n=0}^{\infty} (-)^n \Omega_n \frac{I_{n+\alpha+1/2}(t)}{t^{\alpha+1/2}} P_n^{(\alpha,\alpha)}(x), \tag{5}$$

$$\Omega_n = \frac{2^{1/2-\alpha}\pi^{1/2}(n + \alpha + \tfrac{1}{2})\Gamma(n + 2\alpha + 1)}{\Gamma(n + \alpha + 1)}. \tag{6}$$

Then (4) yields

$$A_n = e^{(n-\alpha-1)[\pi i/2]} \Omega_n \mathcal{H} \left\{ \frac{f(t)}{t^{\alpha+1}} \right\}_{\substack{y=i \\ \nu=n+\alpha+1/2}}, \tag{7}$$

Received December 28, 1959; revised August 4, 1960. This work was supported by the United States Air Force through the Wright Air Development Division.

$$\mathcal{H}\{F(t)\} = \int_0^\infty F(t)J_\nu(yt)(yt)^{1/2}\,dt. \tag{8}$$

$\mathcal{H}\{F(t)\}$ denotes the Hankel transform of $F(t)$ [2].

When $\alpha = -\frac{1}{2}$, (7) furnishes the coefficients for the Chebyshev expansion

$$g(x) = \int_0^\infty e^{-xt}f(t)\,dt = \sum_{n=0}^\infty C_n T_n(x), \qquad -1 \leqq x \leqq 1, \tag{9}$$

where

$$C_n = \epsilon_n\, e^{(n-1/2)[\pi i/2]}\mathcal{H}\left\{\frac{f(t)}{t^{1/2}}\right\}_{\substack{y=i\\ \nu=n}}, \qquad \epsilon_n = \begin{matrix}1, n = 0,\\ 2, n > 0.\end{matrix} \tag{10}$$

If we replace t by it in (5), we find that the same method is applicable when $g(x)$ is a Fourier transform of $f(t)$. We omit details, but the key results for the sine and cosine transforms are as follows.

$$\begin{matrix} g_1(x) \\ g_2(x) \end{matrix} = \int_0^\infty f(t) \begin{matrix}\sin\\ \cos\end{matrix} (xt)\,dt = \sum_{n=0}^\infty \begin{matrix} S_n \\ C_n \end{matrix} P_n^{(\alpha,\alpha)}(x), \qquad -1 \leqq x \leqq 1 \tag{11}$$

where

$$S_n = \begin{cases} 0, & n \text{ even}, \\ e^{(n-1)[\pi i/2]}\Omega_n \mathcal{H}\left\{\dfrac{f(t)}{t^{\alpha+1}}\right\}_{\substack{y=1\\ \nu=n+\alpha+1/2}}, & n \text{ odd}, \end{cases} \tag{12}$$

and

$$C_n = \begin{cases} 0, & n \text{ odd}, \\ e^{n\pi i/2}\Omega_n\, \mathcal{H}\left\{\dfrac{f(t)}{t^{\alpha+1}}\right\}_{\substack{y=1\\ \nu=n+\alpha+1/2}}, & n \text{ even}. \end{cases} \tag{13}$$

3. The Chebyshev Expansion for $1/(y + a)^k$. Let $g(x) = \left[\dfrac{x+1}{2} + a\right]^{-k}$. Then

$$\mathcal{L}^{-1}\{g(x)\} = \frac{2^k}{(k-1)!}\, e^{-(2a+1)t}t^{k-1} = f(t). \tag{14}$$

Use (10) and let $y = \dfrac{x+1}{2}$. Then $T_n(2y - 1) = T_n^*(y), 0 \leqq y \leqq 1$, is the shifted Chebyshev polynomial [3] and

$$\frac{1}{(y+a)^k} = \left\{\sum_{n=0}^\infty \frac{\epsilon_n(-)^n(k+n-1)!}{(k-1)!} P_{k-1}^{-n} \cdot \left[\frac{2a+1}{2\sqrt{a^2+a}}\right] T_n^*(y)\right\} \Big/ (a^2+a)^{k/2} \qquad 0 \leqq y \leqq 1, \qquad a > 0, \tag{15}$$

where $P_\nu^\mu(x)$ is the Legendre function [1, v. 1. p. 120]. For $k = 1$, (15) agrees with a result of Luke [4].

TABLE 1

Coefficients for the Series

$$\psi(x+a) = \sum_{n=0}^{\infty} C_n T_n(x), \qquad \ln \Gamma(x+a) = \sum_{n=0}^{\infty} S_n T_n(x).$$

n	$a = 2$		$a = 3$		$a = 4$		$a = 5$	
	C_n	S_n	C_n	S_n	C_n	S_n	C_n	S_n
0	0.30459199	0.17002422	0.88194225	0.79383494	1.23549564	1.86343494	1.49369453	3.23372482
1	.72037978	.36686678	.41097870	.90276517	.28965835	1.24591092	.22406724	1.49994422
2	−.12454959	.17315258	−.04164582	.10135581	−.02083054	.07191856	−.01249938	.05578533
3	.02776946	−.01962889	.00555546	−.00680364	.00198412	−.00343655	.00092592	−.00207042
4	−.00677624	.00325570	−.00082401	.00067831	−.00021127	.00024503	−.00007686	.00011489
5	.00172388	−.00063281	.00012898	−.00008032	.00002385	−.00002085	.00000678	−.00000762
6	−.00044818	.00013383	−.00002083	.00001046	−.00000279	.00000196	−.00000062	.00000056
7	.00011794	−.00002978	.00000343	−.00000145	.00000033	−.00000020	.00000006	−.00000004
8	−.00003125	.00000685	−.00000057	.00000021	−.00000004	.00000002	−.00000001	—
9	.00000832	−.00000161	.00000010	−.00000003	—	—	—	—
10	−.00000222	.00000039	−.00000002	—	—	—	—	—
11	.00000059	−.00000009	—	—	—	—	—	—
12	−.00000016	.00000002	—	—	—	—	—	—
13	.00000004	−.00000001	—	—	—	—	—	—
14	−.00000001	—	—	—	—	—	—	—
15	—	—	—	—	—	—	—	—

4. The Psi and Log Gamma Functions. These examples show how a property of the Laplace transform may be used to advantage when applying (4) and (8). We know that

$$\mathfrak{L}\{e^{-at}f(t)\} = g(x + a). \tag{16}$$

If $g(x)$ cannot be expanded in symmetric Jacobi polynomials, a in (16) can often be chosen so that $g(x + a)$ has a convergent expansion. Let

$$g(x) = \psi^{(m)}(x) = D^{m+1} \log \Gamma(x). \tag{17}$$

Since $\psi^{(m)}(x)$ has poles at zero and the negative integers, we cannot expand the function over $-1 \leqq x \leqq 1$. However, if

$$g(x) = \psi^{(m)}(x + a), \tag{18}$$

then

$$f(t) = \mathfrak{L}^{-1}\{g(x)\} = (-)^{m+1}e^{-at}t^m[1 - e^{-t}]^{-1}, \tag{19}$$

and if $\mathrm{Re}(a) > 1$, (7), and in particular (10), may be used since (18) is analytic for $|x| \leqq 1$. Substituting (19) in (10) and expanding $(1 - e^{-t})^{-1}$ by the binomial theorem, we have

$$C_n = -\epsilon_n \sum_{k=0}^{\infty} \frac{d^m}{dx^m}\left[\frac{(\sqrt{x^2 - 1} - x)^n}{\sqrt{x^2 - 1}}\right]\Bigg|_{x=k+a}. \tag{20}$$

Setting m equal to zero, we get

$$C_n = -\epsilon_n \sum_{k=0}^{\infty} \frac{[\sqrt{(k+a)^2 - 1} - (k + a)]^n}{\sqrt{(k + a)^2 - 1}}, \qquad n \geqq 1. \tag{21}$$

TABLE 2

Coefficients for the Series

$$Ci(x) = \int_{\infty}^{x} \frac{\cos t}{t}\, dt = \log(x) + \sum_{n=0}^{\infty} A_{2n}T_{2n}\left(\frac{x}{a}\right), \qquad 0 < x \leqq a$$

$$Si(x) = \int_{0}^{x} \frac{\sin t}{t}\, dt = \sum_{n=0}^{\infty} B_{2n+1}T_{2n+1}\left(\frac{x}{a}\right), \qquad -a \leqq x \leqq a$$

n	$a = 2$		$a = 5$	
	A_{2n}	B_{2n+1}	A_{2n}	B_{2n+1}
0	0.13529 62627	1.69809 09708	−0.96313 15550	2.08578 21107
1	−.42327 51922	−.09558 49521	−1.13103 16550	−.67042 59749
2	.01822 27219	.00295 78196	.34661 70891	.15186 68742
3	−.00041 57650	−.00005 14215	−.05698 43620	−.01861 43512
4	.00000 56716	.00000 05642	.00537 47844	.00138 96747
5	−.00000 00511	−.00000 00042	−.00032 52237	−.00006 95137
6	.00000 00003	—	.00001 36729	.00000 24908
7	—	—	−.00000 04226	−.00000 00671
8	—	—	.00000 00100	.00000 00014
9	—	—	−.00000 00002	—

If $n = 0$, (21) diverges, and for $n = 1$ the series is slowly convergent, but since $T_n(1) = 1$, $T_n(-1) = (-)^n$, we may solve for C_0 and C_1 in terms of higher computable coefficients, i.e.,

$$\text{(22)} \qquad \begin{cases} C_0 = \dfrac{\psi(a+1) + \psi(a-1)}{2} - \displaystyle\sum_{k=1}^{\infty} C_{2k}, \\[2ex] C_1 = \dfrac{\psi(a+1) - \psi(a-1)}{2} - \displaystyle\sum_{k=1}^{\infty} C_{2k+1}. \end{cases}$$

Integration of the series defined by (21) yields a Chebyshev expansion for $\ln \Gamma(x + a)$ because [3]

$$\text{(23)} \qquad \int T_n(x)\,dx = \frac{1}{2}\left[\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right] + C.$$

In Table 1 are listed coefficients for the Chebyshev expansions of $\psi(x + a)$ and $\log \Gamma(x + a)$, $a = 2(1)5$, $n = 0(1)15$ to 8D.

5. The Sine and Cosine Integrals. For examples of (11)–(13) let

$$\text{(24)} \qquad \begin{matrix} g_1(x) \\ g_2(x) \end{matrix} = \begin{matrix} (1 - \cos ax)/x \\ \sin ax/x \end{matrix} = \int_0^{\infty} f(t) \begin{matrix} \sin xt \\ \cos xt \end{matrix}\, dt,$$

$$\text{(25)} \qquad f(t) = \begin{cases} 1, & 0 < x < a, \\ 0, & a < x < \infty. \end{cases}$$

Using [2, v. 2, p. 333, No. (1)] to evaluate (12) and (13) for $\alpha = -\frac{1}{2}$, we find that

$$\text{(26)} \qquad S_n = \begin{cases} 0, & n \text{ even}, \\ 4e^{(n-1)[\pi i/2]} \displaystyle\sum_{k=0}^{\infty} J_{n+2k+1}(a), & n \text{ odd}, \end{cases}$$

$$\text{(27)} \qquad C_n = \begin{cases} 0, & n \text{ odd}, \\ 2\epsilon_n e^{n\pi i/2} \displaystyle\sum_{k=0}^{\infty} J_{n+2k+1}(a), & n \text{ even}. \end{cases}$$

Let $a = 2$ and 5 in (26) and (27), and use [1, v. 2, p. 145, No. (6)] and (23) to obtain the expansion whose coefficients are listed in Table 2.

The author wishes to thank Yudell L. Luke and also Charles Himmelberg and Jerry Fields for their helpful suggestions during the preparation of this paper.

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On Stability Criteria of Explicit Difference Schemes for Certain Heat Conduction Problems with Uncommon Boundary Conditions

By Arnold N. Lowan

Abstract. Stability criteria are derived for the explicit difference schemes appropriate to the following problems: 1) heat conduction in a slab in contact with a well stirred liquid; 2) heat conduction in a slab radiating to one face of a thin slab with infinite thermal conductivity, the other face of which radiates into a medium at prescribed temperature; 3) heat conduction in a cylinder radiating to the inner surface of a thin coaxial cylindrical shell with infinite thermal conductivity, the outer surface of which radiates into a medium at prescribed temperature.

Although the exact analytical solutions of certain problems in heat conduction involving complicated boundary conditions are known, the complexity of the analytical expressions is often such as to make them impractical for the numerical evaluation of the solutions. This is for instance the case of the writer's solution of the problem of "heat conduction in a solid in contact with a well stirred liquid" [1]. It is also the case of the problems treated by Walter P. Reid and dealing with the heat conduction in a semi-infinite solid (or cylinder) when the boundary surface radiates to one boundary surface of a thin slab (or thin cylindrical shell), with infinite thermal conductivity, the other boundary surface of which radiates into a medium at prescribed temperature [2], [3].

To obtain numerical answers to the above problems it is expedient to evaluate the solutions of the appropriate explicit difference analogs. The object of this report is to derive the stability criteria for the difference schemes appropriate to the problems above-mentioned.

Consider first the problem of heat conduction in a slab one face of which is in contact with a well-stirred liquid. For the sake of concreteness we shall first assume that the other face is kept at 0°C. The mathematical formulation of the problem is as follows:

$$(1) \qquad \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \qquad 0 \leqq x \leqq a, \quad t > 0$$

$$(2) \qquad T(x, 0) = f(x)$$

$$(3\text{A}) \qquad T(0, t) = 0$$

$$(4) \qquad -\frac{\partial T}{\partial x} = \left(\frac{\rho_0 c_0 d}{K}\right)\frac{\partial T}{\partial t} = \sigma\frac{\partial T}{\partial t} \quad \text{(say) for} \quad x = a.$$

In (4) we have the boundary condition appropriate to the case where the face $x = a$ is in contact with a layer of a well-stirred liquid of width d, density ρ_0 and specific heat c_0. The constants K and k are the thermal conductivity and thermal diffusivity of the slab, respectively.

Received August 17, 1959; revised November 4, 1960.

We shall consider the problem of stability of the explicit difference scheme

$$T_{m,n+1} = r\,T_{m-1,n} + (1-2r)T_{m,n} + r\,T_{m+1,n}\,; \quad m = 1, 2, 3, \cdots M \tag{5}$$

where $T_{m,n} = T(m\Delta x, n\Delta t)$; $r = k\Delta t/(\Delta x)^2$, and $a = (M+1)\Delta x$. The difference analog of (4) is

$$\frac{T_{M,n} - T_{M+1,n}}{\Delta x} = \sigma\,\frac{T_{M+1,n+1} - T_{M+1,n}}{\Delta t} \tag{6}$$

whence

$$T_{M+1,n+1} = sT_{M,n} + (1-s)T_{M+1,n}\,; \qquad s = \frac{\Delta t}{\sigma\Delta x}. \tag{7}$$

Since $\Delta t = r(\Delta x)^2/k = r(\Delta x)^2\rho c/K$ where ρ and c are the density and specific heat of the slab, it follows that $s = r(\Delta x/d)(\rho c/\rho_0 c_0)$. Since usually $\Delta x \ll d$ and $c \ll c_0$ it is reasonable to assume that $\Delta x/d\cdot\rho c/\rho_0 c_0 < 1$ and *a fortiori* $s < 1$, if we anticipate the fact that the criterion for the stability of the explicit difference scheme under consideration is $r \leqq \frac{1}{2}$.

The system of $M + 1$ equations consisting of (5) and (7) may be written in the matrix-vector form

$$\mathbf{T}_{n+1} = A\,\mathbf{T}_n \tag{8}$$

where $\mathbf{T}_n$ and $\mathbf{T}_{n+1}$ are the $(M+1)$ dimensional vectors whose components are the temperatures $T_{m,n}$ and $T_{m,n+1}$, respectively, with $m = 1, 2, 3, \cdots M+1$ and where

$$A = \begin{bmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & r & 1-2r & r \\ & & & s & 1-s \end{bmatrix} \tag{9}$$

Examination of the matrix A shows that if $r \leqq \frac{1}{2}$ so that $1 - 2r \geqq 0$, then the largest of the sums of the absolute values of the elements of each of the first M rows of A is equal to 1. Furthermore, we have seen that it is reasonable to assume that $s \leqq 1$ or $1 - s \geqq 0$; accordingly, the sum of the absolute values of the elements of the last row of A is again $= 1$. The condition for the stability of the difference scheme under consideration is thus satisfied [4].

In the above treatment we assumed that the face $x = 0$ is kept at 0°C. If instead the temperature at $x = 0$ is prescribed, so that

$$T(0, t) = \phi(t) \tag{3B}$$

it is readily seen that the stability criterion $r \leqq \frac{1}{2}$ we found above is valid, since the error vector $\mathbf{E}_n$ satisfies the difference equation (8) and since it vanishes on $x = 0$. We define the error vector $\mathbf{E}_n$ as the vector $\mathbf{T}_n - \mathbf{T}_n^*$, where $\mathbf{T}_n$ is the "true" solution of (8) and $\mathbf{T}_n^*$ is the vector generated by successive application of

(8) when during some time step an inaccurate vector $\mathbf{T}_k^*$ (where k can of course be $= 0$) is used in lieu of the true $\mathbf{T}_k$. It may be briefly mentioned that the above stability criterion is equally valid in case of boundary conditions of the form

$$\frac{\partial T}{\partial x} = 0 \quad \text{for} \quad x = 0 \tag{3C}$$

$$\frac{\partial T}{\partial x} = hT \quad \text{for} \quad x = 0. \tag{3D}$$

In the case of the boundary condition (3C) the first element of the first row of the matrix A is replaced by $1 - r$; in the case of the boundary condition (3D) the element in question is replaced by $1 - 2r + \alpha r$ where $\alpha = 1/(1 + h\Delta x)$. In either case the largest of the sums of the absolute values of the elements of the rows of A is still equal to one and therefore the stability condition is satisfied if $r \leq \frac{1}{2}$. For a discussion of convergence the reader is referred to [4], Section VI.

Consider now the problem of heat conduction in a slab subject to the conditions stipulated in the first of Reid's articles mentioned above. The mathematical formulation of the problem is as follows:

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \qquad 0 \leq x \leq a, t > 0 \tag{10}$$

$$T(x, 0) = f(x) \tag{11}$$

$$T(0, t) = 0 \tag{12}$$

$$-K\left(\frac{\partial T}{\partial x}\right)_{x=a} = h_1\,[T(a, t) - v(t)] \tag{13}$$

$$\rho_0\, c_0\, d\, \frac{\partial v}{\partial t} = h_1\,[T(a, t) - v(t)] - h_2\, v(t) \tag{14}$$

$$v(0) = V. \tag{15}$$

In the above:

K = thermal conductivity of the thick slab
k = thermal diffusivity of the thick slab
h_1 = coefficient of heat transfer between the two slabs
h_2 = coefficient of heat transfer between the thin slab and the surrounding medium (with temperature zero)
$v(t)$ = temperature of thin slab
ρ_0, c_0 and d are the density, specific heat and width of the thin slab.

If we put

$$\frac{h_1}{K} = b; \qquad \frac{h_2}{K} = c; \qquad \frac{\rho_0\, c_0\, d}{K} = \sigma \tag{16}$$

equations (13) and (14) assume the form

$$-\left(\frac{\partial T}{\partial x}\right)_{x=a} = b\,[T(a, t) - v(t)] \tag{13*}$$

$$\sigma \frac{\partial v}{\partial t} = b[T(a, t) - v(t)] - cv(t). \tag{14*}$$

We shall investigate the stability of the explicit difference scheme

$$\frac{T_{m,n+1} - T_{m,n}}{\Delta t} = \frac{k}{(\Delta x)^2}(T_{m-1,n} - 2T_{m,n} + T_{m+1,n})$$

or

$$(17) \qquad T_{m,n+1} = rT_{m-1,n} + (1 - 2r)T_{m,n} + rT_{m+1,n}, \qquad m = 1, 2, 3, \cdots M$$

where

$$T_{m,n} = T(m\Delta x, n\Delta t), \qquad r = \frac{k\Delta t}{(\Delta x)^2} \quad \text{and} \quad \Delta x = \frac{a}{M+1}.$$

The difference analogs of (13*) and (14*) are

$$\frac{T_{M,n} - T_{M+1,n}}{\Delta x} = b(T_{M+1,n} - v_n)$$

or

$$(18) \qquad T_{M+1,n} = \frac{1}{1 + b\Delta x} T_{M,n} + \frac{b\Delta x}{1 + b\Delta x} v_n = pT_{M,n} + qv_n \quad \text{(say)}$$

and

$$\sigma \frac{v_{n+1} - v_n}{\Delta t} = bT_{M+1,n} - (b + c)v_n$$

or

$$(19) \qquad v_{n+1} = \frac{b\Delta t}{\sigma} T_{M+1,n} + \left[1 - \frac{(b+c)\,\Delta t}{\sigma}\right] v_n = \alpha T_{M+1,n} + \beta v_n \quad \text{(say)}$$

where we have written v_n and v_{n+1} for $v(n\Delta t)$ and $v[(n+1)\Delta t]$, respectively. If from (18) and (19) we eliminate $T_{M+1,n}$ we get

$$(20) \qquad v_{n+1} = \alpha p t_{M,n} + (\beta + \alpha q)v_n.$$

If we rewrite (18) in the form

$$(18^*) \qquad T_{M+1,n+1} = pT_{M,n+1} + qv_{n+1}$$

and eliminate v_n and v_{n+1} from (18*), (18), and (19), we get

$$(21) \qquad T_{M+1,n+1} = (\beta + \alpha q)T_{M+1,n} + pT_{M,n+1} - p\beta T_{M,n}.$$

If in (21) we replace $T_{M,n+1}$ by its expression from (17) with $h = M$, we ultimately get

$$(22) \qquad \begin{aligned} \mathrm{T}_{M+1,n+1} &= prT_{M-1,n} + p(1 - 2r - \beta)T_{M,n} + (pr + \beta + \alpha q)T_{M+1,n} \\ &= PT_{M-1,n} + QT_{M,n} + ST_{M+1,n} \quad \text{(say)}. \end{aligned}$$

Starting with the values of $T_{m,n}$ for $n = 0$ equations (17) and (22) will generate in succession the values of $T_{m,n}$ for $n = 1, 2, \cdots$ and $m = 1, 2, 3, \cdots M + 1$. Similarly, starting with $v(t) = V$ for $t = 0$ equation (19) will generate in succession the values of $v_n = v(n\Delta t)$.

The system of $M + 1$ equations consisting of (17) and (22) with the errors $E_{m,n}$ written for the $T_{m,n}$'s may be written in the matrix-vector form

$$\mathbf{E}_{n+1} = A\mathbf{E}_n \tag{23}$$

where $\mathbf{E}_n$ and $\mathbf{E}_{n+1}$ are the $(M + 1)$ dimensional vectors whose components are the errors in the values of the temperature at the $M + 1$ lattice points $m\Delta x$, $m = 1, 2, 3, \cdots M + 1$ at the times $n\Delta t$ and $(n + 1)\Delta t$ and where

$$A = \begin{bmatrix} 1-2r & r & & & & \\ r & 1-2r & r & & & \\ & r & 1-2r & r & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & r & 1-2r & r \\ & & & P & Q & S \end{bmatrix} \tag{24}$$

If we assume

$$0 < r \leqq \tfrac{1}{2} \tag{25}$$

it is clear from (24) that the sum of the absolute values of the elements of the first M rows of A is equal to 1. If in addition we assume that

$$1 - 2r - \beta \geqq 0, \qquad pr + \beta + \alpha q \geqq 0 \tag{26}$$

then, it is seen from (22) that

$$|P| + |Q| + |S| = p - p\beta + \beta + \alpha q = 1 - q\frac{c\Delta t}{\sigma} < 1. \tag{27}$$

Thus, if (25) and (26) are satisfied, the largest of the sums of absolute values of the elements of the rows of matrix A is equal to 1, and therefore the difference scheme under consideration is stable.

The inequalities (25) and (26) may be written

$$\frac{n\Delta t}{(\Delta x)^2} \leqq \frac{1}{2}$$

$$2\,\frac{n\Delta t}{(\Delta x)^2} \leqq \frac{(b + c)\Delta t}{\sigma}, \qquad \left[r + \frac{b^2\Delta x\Delta t}{\sigma}\right]\frac{1}{1 + b\Delta x} + \left[1 - \frac{b + c}{\sigma}\,\Delta t\right] \geqq 0$$

whence

$$(\Delta x^2) \geqq \frac{2n\sigma}{b + c} \tag{28}$$

$$\Delta t \leqq \min\left[\frac{\Delta x^2}{2n}, \frac{\sigma}{b + c}\right]. \tag{29}$$

In conclusion, choosing the intervals Δx and Δt in accordance with (28) and (29) will insure the stability of the difference scheme under consideration.

Finally, consider the problem of heat conduction in a cylinder subject to the boundary conditions stipulated in the second of Reid's articles mentioned above.

If, for the sake of convenience, we denote by x the radial distance ordinarily denoted by r, the mathematical formulation of the problem is as follows:

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{1}{x}\frac{\partial T}{\partial x}\right) \tag{30}$$

$$T(x, 0) = f(x) \tag{31}$$

$$\frac{\partial T}{\partial x} = 0 \quad \text{for} \quad x = 0 \tag{32}$$

$$-K\left(\frac{\partial T}{\partial x}\right)_{x=a} = m_1\,[T(a, t) - v(t)] \tag{33}$$

$$2\pi\rho_0\, c_0\, d\,\frac{\partial v}{\partial t} = m_1\,[T(a, t) - v(t)] - m_2\, v(t)\,. \tag{34}$$

The only essential difference between the above problem for the cylinder and the corresponding problem for the slab is the differential equation (30) which takes the place of (10) and the new condition (32). The difference analogs of (30) and (32) are

$$\frac{T_{m,n+1} - T_{m,n}}{\Delta t} = k\left[\frac{T_{m-1,n} - 2T_{m,n} + T_{m,n+1}}{(\Delta x)^2} + \frac{1}{m\Delta x}\cdot\frac{T_{m+1,n} - T_{m-1,n}}{2\Delta x}\right]$$

or

$$T_{m,n+1} = r\left(1 - \frac{1}{2m}\right)T_{m-1,n} + (1 - 2r)T_{m,n} + r\left(1 + \frac{1}{2m}\right)T_{m+1,n} \tag{35}$$

$$m = 1, 2, 3, \cdots M$$

and

$$T_{0,n} = T_{1,n}\,. \tag{36}$$

The difference equation (35) takes the place of the difference equation (17). In view of (35) and (36) it is readily seen that the previous developments for the slab may be carried out with the sole exception that in lieu of (24) the matrix A is now

$$A = \begin{bmatrix} 1-2r & \frac{3}{2}r & & & & \\ \left(1-\frac{1}{4}\right)r & 1-2r & \left(1+\frac{1}{4}\right)r & & & \\ & \left(1-\frac{1}{6}\right)r & 1-2r & \left(1+\frac{1}{6}\right)r & & \\ & \cdots\cdots & \cdots\cdots & \cdots\cdots & & \\ & & \left(1-\frac{1}{2M}\right)r & 1-2r & \left(1+\frac{1}{2M}\right)r \\ & & P & Q & S \end{bmatrix} \tag{37}$$

where the symbols P, Q, and S have the same significance as before. Since for the first M rows of A the largest of the sums of the absolute values of the elements is

equal to 1 (provided $r \leqq \frac{1}{2}$) we reach the conclusion that the criteria of stability are identical with those of the previous problem and are given in the inequalities (28) and (29).

It may be briefly mentioned that the above analysis of stability may be readily extended to the case where the cylinder and the coaxial thin cylindrical shell are replaced by a sphere and a concentric thin spherical shell.

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On Numbers of the Form $n^4 + 1$

By Daniel Shanks

1. The Number of Primes. Let $Q_1(N)$ be the number of primes of the form $n^4 + 1$ for $1 \leqq n \leqq N$. By a double sieve argument similar to that used for primes of the form $n^2 + a$, [1], and for Gaussian twin primes, [2], one is led to the following conjecture:

$$Q_1(N) \sim \frac{1}{4} s_1 \int_2^N \frac{dn}{\log n} \tag{1}$$

where

$$s_1 = \prod_{p=3}^{\infty} \left[1 - \frac{\left(\frac{-1}{p}\right) + \left(\frac{2}{p}\right) + \left(\frac{-2}{p}\right)}{p - 1} \right], \tag{2}$$

the product being taken over all odd primes with $\left(\frac{a}{p}\right)$ the Legendre symbol. Now

$$\frac{s_1 L_1(1) L_2(1) L_{-2}(1)}{\zeta_1^2(2)} = \prod_{p=8m+1} \left(1 - \frac{4}{p}\right) \left(\frac{p+1}{p-1}\right)^2 \tag{3}$$

where this product is taken over all primes of the form $8m + 1$ and $L_a(s)$ and $\zeta_a(s)$ are as defined in [1, p. 323]. We may therefore write

$$s_1 = \frac{\pi^2}{4 \log (1 + \sqrt{2})} \prod_{p=8m+1} \left(1 - \frac{4}{p}\right) \left(\frac{p+1}{p-1}\right)^2. \tag{4}$$

To evaluate this slowly convergent product we use the identity

$$1 - 4x = \left(\frac{1-x}{1+x}\right)^2 \left(\frac{1-x^2}{1+x^2}\right)^4 \left(\frac{1-x^3}{1+x^3}\right)^{10} \left(\frac{1-x^4}{1+x^4}\right)^{32} \cdots, \tag{5}$$

which is valid for $x < \frac{1}{4}$, and the identity

$$\frac{\zeta_1^2(2s)}{\zeta_1(s) L_1(s) L_2(s) L_{-2}(s)} = \prod_{p=8m+1} \left(\frac{p^s - 1}{p^s + 1}\right)^2, \tag{6}$$

which is valid for $s > 1$. From tables of $\zeta_a(s)$ and $L_a(s)$ we thus obtain

$$s_1 = 2.67896 \cdots \tag{7}$$

and therefore

$$Q_1(N) \sim \bar{Q}_1(N) = 0.66974 \int_2^N \frac{dn}{\log n}. \tag{8}$$

It is interesting to compare this formula with that for the conjectured number [1] of primes of the form $n^2 + 1$,

$$P_1(N) \sim \bar{P}_1(N) = 0.68641 \int_2^N \frac{dn}{\log n}. \tag{9}$$

Received November 1, 1960.

TABLE 1

N	$Q_1(N)$	$\bar{Q}_1(N)$	$Q_1/\bar{Q}_1$
100	18	19.5	0.924
200	30	32.9	0.911
300	44	45.1	0.976
400	52	56.5	0.920
500	63	67.5	0.934
600	75	78.1	0.960
700	80	88.4	0.905
800	94	98.6	0.954
900	98	108.5	0.903
1000	109	118.3	0.922

The coefficients are nearly equal and have analogous formulae:

$$(10)\qquad \begin{aligned} 0.68641 &= \frac{1}{2}\prod_{p=3}^{\infty}\left[1 - \frac{\left(\frac{-1}{p}\right)}{p-1}\right] \\ 0.66974 &= \frac{1}{4}\prod_{p=3}^{\infty}\left[1 - \frac{\left(\frac{-1}{p}\right) + \left(\frac{2}{p}\right) + \left(\frac{-2}{p}\right)}{p-1}\right]. \end{aligned}$$

2. A Table. A comparison of $\bar{Q}_1(N)$ with the actual counts $Q_1(N)$ is handicapped by the very rapid increase in $n^4 + 1$. The 109th prime is already 984 095 744 257, nearly a trillion. A. Gloden [3] has completed the factorization of all $n^4 + 1$ up to $n = 1000$, following the work of Cunningham and others. He has kindly counted the primes for us, where $400 < n \leqq 1000$, and using his results we present Table 1. The deviations of $Q_1/\bar{Q}_1$ from unity are not unduly large considering the relatively small upper limit for N. For $P_1(N)$ and for the ordinary prime count $\pi(N)$ we have similar deviations for $N = 1000$; $\pi(1000)/li(1000) = 0.951$ and $P_1(1000)/\bar{P}_1(1000) = 0.924$.

3. Four Classes of Numbers. When we consider that Euler determined $P_1(N)$ up to $N = 1500$ over two hundred years ago [4], the present table of $Q_1(N)$ up to $N = 1000$ seems rather meager. The much greater difficulty of factoring the $n^4 + 1$ numbers is fundamentally due to their much greater magnitude—but there are interesting technical differences also. The sieve method for $n^4 + 1$ used by Gloden, Cunningham, and others has three phases.

A. Compile a list of primes of the form $8m + 1$

B. For each such prime solve the congruence

$$\begin{cases} x^4 \equiv -1 \pmod{p} \\ x < p \end{cases}$$

for its four roots. (Given one solution x_1 the remaining three are congruent to $-x_1$, x_1^3, and $-x_1^3$.)

C. With each x and each p divide out a factor of p for each $n = x_i + mp$. Similarly determine those $n^4 + 1$ divisible by p^2, p^3, etc.

Now unfortunately there is much waste computation here. For instance, the hundred $n^4 + 1$ for $n \leqq 100$ have 122 different primes of the form $8m + 1$ as factors. Yet *all* 295 of the $8m + 1$ primes $<100^2$ must be examined in phases A and B, since *a priori* any such prime *may* be a factor of the $n^4 + 1$. And clearly this waste increases rapidly with N,—for $N = 1000$ we must examine all 19552 of the $8m + 1$ primes $<1000^2$ to factor out the (approximately) 1300 distinct actual prime factors.

On the contrary, in the author's sieve [5] for $n^2 + 1$ there is no waste computation and no phases A and B, either. The primes arise automatically in the sieve itself, together with the corresponding solutions of the congruence, $x^2 \equiv -1 \pmod{p}$.

This significant difference comes about as follows. For every n, $n^2 + 1$ either has no new prime factor (n is "reducible") or it has precisely one new prime factor—and that to the first power (n is "irreducible"). Therefore, if all prime factors corresponding to smaller values of n have already been sieved out, each new prime stands exposed at the smallest n which satisfies $n^2 \equiv -1 \pmod{p}$. But for $n^4 + 1$ we have not two but *four* classes of n; there are either 0, 1, 2, or 3 new prime factors in $n^4 + 1$. It is the occurrence of the "double" and "triple" irreducibles (i.e., 2 and 3 new primes) which prevents the use of the *automatic*, $n^2 + 1$ type sieve for $n^4 + 1$. Already for $n = 10$ we have a double irreducible

$$10^4 + 1 = 73 \cdot 137,$$

with the two new primes 73 and 137.

Let $R(N)$, $I_1(N)$, $I_2(N)$ and $I_3(N)$ be the number of "reducibles" (no new prime) and single, double, and triple irreducibles respectively which are $\leqq N$. For example, $I_1(120) = 92$ and $I_2(120) = 28$. Further, $R(120) = I_3(120) = 0$, since neither reducibles nor triple irreducibles arise for $n \leqq 120$. For larger n (from Gloden's tables) we find both reducibles

$$29588^4 + 1 = 17^2 \cdot 41 \cdot 113 \cdot 1249 \cdot 16073 \cdot 28513$$

and triple irreducibles

$$23762^4 + 1 = 637489 \cdot 693569 \cdot 721057,$$

but they are rare.

The mean number of new primes is

$$\nu(N) = \frac{I_1(N) + 2I_2(N) + 3I_3(N)}{N}, \tag{11}$$

and in analogy with the situation for $n^2 + 1$ the question arises whether $\nu(N)$ has a limit for $N \to \infty$. For $n^2 + 1$, John Todd [5, p. 83] has conjectured $\nu(N) \to \log 2 = 0.693$. For $n^4 + 1$ and a modest N we have $\nu(N) \approx 1.3$. Analogy with Todd's results concerning $n^2 + 1$ and $\log 2$ would suggest a limit of $\log 4$ for $n^4 + 1$, but there is no serious evidence in favor of this.

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1. Daniel Shanks, "On the conjecture of Hardy and Littlewood concerning the number of primes of the form $n^2 + a$," *Math. Comp.*, v. 14, 1960, p. 321–332.

2. Daniel Shanks, "A note on Gaussian twin primes," *Math. Comp.*, v. 14, 1960, p. 201–203.

3. A. Gloden, "A note on factors of $n^4 + 1$," *Math. Comp.*, v. 14, 1960, p. 278–279. Also see RMT 42, *Math. Comp.*, v. 14, 1960, p. 284. For earlier bibliography see RMT 109, *MTAC*, v. 11, 1957, p. 274, and RMT 2, *MTAC*, v. 12, 1958, p. 63.

4. L. E. Dickson, *History of the Theory of Numbers*, Stechert, New York, 1934, v. 1, p. 381. According to Dickson, Euler (1752) gave $P_1(1500) = 161$, which is correct, and $Q_1(34) = 8$, which is incorrect—he omits the prime $28^4 + 1$.

5. Daniel Shanks, "A sieve method of factoring numbers of the form $n^2 + 1$," *MTAC*, v. 13, 1959, p. 78–86.

TECHNICAL NOTES AND SHORT PAPERS

On Modular Computation

By Henry B. Mann

It will be assumed that the reader is familiar with the elementary theory of congruences [1].

Let $m_1, m_2, \cdots, m_s$ be s integers relatively prime in pairs and let $M = m_1 m_2 \cdots m_s$. Let $x_1, x_2, \cdots, x_s$ be an ordered set of s integers such that $0 \leqq x_i < m_i$. There exists one and only one residue class x mod M such that $x \equiv x_i \pmod{m_i}$ and we therefore write $x = (x_1, x_2, \cdots, x_s)$. If $x = (x_1, \cdots, x_s)$, $y = (y_1, \cdots, y_s)$ then $x \pm y = (x_1 \pm y_1, \cdots, x_s \pm y_s)$, $xy = (x_1y_1, \cdots, x_sy_s)$, where the ith coordinates must be reduced mod m_i. The symbols $(x_1, \cdots, x_s)$ are called modular numbers, but it should be kept in mind that $(x_1, \cdots, x_s)$ does not denote a number but a residue class mod M.

The purpose of this note is to describe a simple iterative procedure to determine the least non-negative residue mod M of a given residue class $(x_1, \cdots, x_s)$.

The iteration process described below gives the least non-negative residue in mixed radix representation.

Notation. The moduli are denoted by $m_1, \cdots, m_s$. We put $m_0 = 1$,

$$M_i = (0, \cdots, 0, 1, 0, \cdots, 0); \qquad i = 1, \cdots, s$$

where 1 occurs in the ith position,

$$\pi_i = m_0 m_1 \cdots m_{i-1} \qquad i = 1, \cdots, s,$$

$[x]_M$ denotes the least non-negative residue corresponding to the modular number $x = (x_1, \cdots, x_s)$.

The representation

$$[x]_M = \sum_{i=1}^{s-1} a_i \pi_i, \qquad 0 \leqq a_i < m_i, \quad i = 1, \cdots, (s-1)$$

is the mixed radix representation of $[x]_M$. Because of the condition $0 \leqq a_i < m_i$, the mixed radix representation of a number a can be coded as a modular number $x_a = (a_1, \cdots, a_s)$.

Now proceed as follows:

Find

$$M_i = \sum_j a_{ij} \pi_j, \qquad \pi_i \equiv \sum x_{ij} M_j \pmod{M}.$$

The jth columns of the matrices (a_{ij}) and (x_{ij}) are residues mod m_j, so that the rows of these matrices may be regarded as modular numbers.

In forming $(x_1, \cdots, x_s)(a_{ij})$ we proceed as in ordinary matrix multiplication but compute the inner product $x_1 a_{1j} + \cdots + x_s a_{sj}$ mod m_j. Similarly, in forming $(a_1, \cdots, a_s)(x_{ij})$ we compute the inner product $a_1 x_{1j} + \cdots + a_s x_{sj}$ mod m_j. Note

Received June 2, 1960.

that if $[x]_M = \sum_j a_j \pi_j$, then $(a_1, \cdots, a_s)(x_{ij}) = x$, but $x(a_{ij})$ gives in general not the radix representation of x, but of $(x(a_{ij}))(x_{ij})^*$.

Given $x = (x_1, \cdots, x_s)$, the iteration proceeds as follows:

x	Δx	Δa	a
$x = (x_1, \cdots, x_s)$			$a^{(1)} = x(a_{ij})$
$x^{(1)} = a^{(1)}(x_{ij})$	$x - x^{(1)}$	$(x - x^{(1)})(a_{ij})$	$a^{(2)} = a^{(1)} + \Delta a^{(1)}$
$x^{(\alpha)} = a^{(\alpha)}(x_{ij})$	$x - x^{(\alpha)}$	$(x - x^{(\alpha)})(a_{ij})$	$a^{(\alpha+1)} = a^{(\alpha)} + \Delta a^{(\alpha)}$

The process ends when $\Delta x = 0$.

The first two coordinates of $x^{(1)}$ will coincide with the first two coordinates of x. Thereafter the first $\alpha + 1$ coordinates of $x^{(\alpha)}$ will coincide with the first $(\alpha + 1)$ coordinates of x so that $x^{(s-1)} = x$ and hence $a^{(s-1)}$ gives the mixed radix representation of $[x]_M$, the coordinates of $a^{(s-1)}$ being the digits of this representation. In many cases, however, less than $(s - 1)$ steps will suffice.

Example 1 (ascending order): $m_1 = 2, m_2 = 3, m_3 = 5, m_4 = 7$.

$$(a_{ij}) = \begin{matrix} 1 & 1 & 2 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 \end{matrix} \qquad x_{ij} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 2 \end{matrix}$$

x	Δx	Δa	a
1, 2, 2, 5			1, 2, 1, 0
x_1 = 1, 2, 1, 4	0, 0, 1, 1	0, 0, 1, 1	1, 2, 2, 1
x_2 = 1, 2, 2, 5			

Hence $[x]_{210} = 1 + 2.2 + 2.6 + 1.30 = 47$.

Example 2 (descending order): $m_1 = 7, m_2 = 5, m_3 = 3, m_4 = 2$.

$$a_{ij} = \begin{matrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 0, \\ 0 & 0 & 0 & 1 \end{matrix} \qquad x_{ij} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1. \\ 0 & 0 & 0 & 1 \end{matrix}$$

x	Δx	Δa	a
4, 3, 1, 0			4, 2, 2, 1
4, 3, 1, 1	0, 0, 0, 1	0, 0, 0, 1	4, 2, 2, 0
4, 3, 1, 0			

Hence $[4, 3, 1, 0]_{210} = 4 + 2.7 + 2.35 = 88$.

Remarks: (1) Arranging the modules in ascending order may have an advantage because the matrices (a_{ij}) and (x_{ij}) are half diagonal and if $0 \leqq y_j < m_j$ then also $0 \leqq y_j < m_{j+1}$ so that y_j never has to be converted. However, any arrangement will work.

(2) No conversion to decimals is necessary to decide if $[x^{(1)}]_M > [x^{(2)}]_M$. If $[x^{(1)}]_M = \sum_j a_j^{(1)} \pi_j$, $[x^{(2)}]_M = \sum_j a_j^{(2)} \pi_j$, then $[x^{(1)}]_M > [x^{(2)}]_M$ if and only if $a_s^{(1)} > a_s^{(2)}$ or $a_s^{(1)} = a_s^{(2)}, \cdots, a_{s-j+1}^{(1)} = a_{s-j+1}^{(2)}, a_{s-j}^{(1)} > a_{s-j}^{(2)}$.

* Caution: The matrix multiplication defined above is not associative.

(3) All calculations are mod m_i so that the digits of the mixed radix representation of $[x]_M$ can be obtained using only calculations mod m_i.

(4) The value $x - x^{(\alpha)}$ can be obtained from $x - x^{(\alpha)} = x - x^{(\alpha-1)} + x^{(\alpha-1)} - x^{(\alpha)}$ so that the modular number x need not be remembered during the whole process.

(5) The matrices (a_{ij}) and (x_{ij}) are computed preliminary to the iteration procedure and are not part of it.

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Generation of Permutations by Transposition

By Mark B. Wells

1. Introduction. As discussed by Tompkins [1], many problems require the generation of all $n!$ permutations of n marks (henceforth called arrangements). This note presents a generation scheme whereby each step consists of merely transposing two of the marks. The bookkeeping is quite simple, thus this scheme is somewhat faster than either the usual dictionary order method or the Tompkins-Paige method [1]. Also, the important property of leaving the $(j + 1)$st position alone until all $j!$ arrangements of the marks in the first j positions have been generated is preserved.

2. Notation. An arrangement of n marks will be given by an n-tuple, $(m_1, m_2, \cdots m_n)$. A permutation, that is, an operation of permuting an arrangement of marks, will be given in cyclic form, with P's modified by subscripts as entries. The subscripts indicate the position of the marks to be moved in the n-tuple on which the permutation is operating. For example, if $a = (1, 2, 5, 4, 3)$ is an arrangement of five marks and $\rho = (P_1P_3P_2)(P_4P_5)$ is a permutation, then $\rho(a) = (2, 5, 1, 3, 4)$.

The bookkeeping for this generation scheme is handled, as in most schemes of this type, by an ordered set of indices t_k, $k = 2, 3, \cdots, n$, where each t_k assumes the values 1 through k and indicates the progress of the subgeneration of the arrangements of marks in positions 1 to k. (This is essentially the "signature" discussed in [1].) Thus there are $n!$ sets of values for the t_k's, one set for each arrangement of the n marks. The set $t_k = 1$ for all k corresponds to the initial arrangement, and successive sets are formed in dictionary order (assuming increasing significance with increasing subscript). An index k' gives at each step the smallest subscript k for which $t_k \neq k$.

3. The Generation Rules. The transposition required at each step depends on the current value of the index k' and on the corresponding value of $t_{k'+1}$(t_{n+1} is assumed $= 1$). The rules are:

I. If k' is even, then interchange the marks in positions k' and $k' - 1$.

Received August 12, 1960.

II. a. If k' is odd and $t_{k'+1} \leqq 2$ then interchange the marks in positions k' and $k' - 1$.

b. If k' is odd and $2 < t_{k'+1} < k'$, then interchange the marks in positions k' and $k' - t_{k'+1} + 1$.

c. If k' is odd and $t_{k'+1} \geqq k$, then interchange the marks in positions k' and 1.

Before proving that these rules yield all $n!$ arrangements in $n! - 1$ applications (starting with a given arrangement), let us illustrate their application for $n = 5$. The rules apply at step s to yield the arrangement given at step $s + 1$.

Step	t_2	t_3	t_4	t_5	t_6	k'	Arrangement 1 2 3 4 5
1	1	1	1	1	1	2	(1, 2, 3, 4, 5)
2	2	1	1	1	1	3	(2, 1, 3, 4, 5)
3	1	2	1	1	1	2	(2, 3, 1, 4, 5)
4	2	2	1	1	1	3	(3, 2, 1, 4, 5)
5	1	3	1	1	1	2	(3, 1, 2, 4, 5)
6	2	3	1	1	1	4	(1, 3, 2, 4, 5)
7	1	1	2	1	1	2	(1, 3, 4, 2, 5)
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	2	3	2	1	1	4	(1, 4, 3, 2, 5)
13	1	1	3	1	1	2	(1, 4, 2, 3, 5)
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
18	2	3	3	1	1	4	(2, 4, 1, 3, 5)
19	1	1	4	1	1	2	(2, 4, 3, 1, 5)
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
24	2	3	4	1	1	5	(3, 4, 2, 1, 5)
25	1	1	1	2	1	2	(3, 4, 2, 5, 1)
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
48	2	3	4	2	1	5	(2, 5, 4, 3, 1)
49	1	1	1	3	1	2	(2, 5, 4, 1, 3)
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
72	2	3	4	3	1	5	(4, 1, 5, 2, 3)
73	1	1	1	4	1	2	(4, 1, 5, 3, 2)
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
96	2	3	4	4	1	5	(5, 3, 1, 4, 2)
97	1	1	1	5	1	2	(5, 3, 1, 2, 4)
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
120	2	3	4	5	1	6	(1, 2, 3, 5, 4)

A close inspection of the above example will reveal the mechanism at work. Following a transposition (P_iP_k) with $i < k$ all $(k - 1)!$ arrangements involving change only in positions 1 through $k - 1$ are generated before P_k appears again. During a complete subgeneration of the $k!$ arrangements of the k leftmost positions, the transposition (P_iP_k), for some particular $i < k$, occurs $k - 1$ times, each time $k' = k$. The particular value of i will be $k - 1$, $k - t_{k+1} + 1$, or 1, according to the rule in force. To insure that no duplicate arrangements appear, the mark initially (at the time the subgeneration begins) in position k and the marks successively (each time (P_iP_k) is performed) in position i must all be distinct. This is accomplished in two ways according as k is even or odd. For an example with $k = 4$, compare the marks in position 4 at step 1, and in position 3 at steps 6, 12, and 18.

LEMMA 1. *Let* $\alpha = (P_{i_1}P_{i_2} \cdots P_{i_{k-1}})$ *with* $i_1, i_2, \cdots, i_{k-1} \leqq k - 1$ *be a cycle and let* $j < k$. *Then* $\alpha[(P_jP_k)\alpha]^{k-1} = (P_jP_k)$.

Proof. This is verified by direct permutation multiplication.

The significance of this lemma is the following. Let k be odd and consider any subgeneration of $k!$ arrangements of the marks in the k leftmost positions. During this subgeneration k' will be equal to k $k - 1$ times, and we will have $k - 1$ identical applications of rule II interspersed with k identical permutations of the first $k - 1$ positions. If, as Lemma 2 will show, this permutation is a cycle, then Lemma 1 says the effect of the entire subgeneration was as a single application of rule II on the initial arrangement.

LEMMA 2. *For k even,* $(P_1P_{k-1})(P_{k-1}P_k)\,[\prod_{i=1}^{k-2} (P_iP_{k-1})(P_{k-1}P_k)](P_{k-2}P_{k-1}) = \rho_k$, *a single cycle, where* $\rho_2 = (P_1P_2)$, $\rho_4 = (P_1P_4P_2P_3)$ *and in general,* $\rho_k = (P_1P_kP_{k-2}\{P_{k-3}P_{k-5} \cdots P_3\}P_{k-1}\{P_{k-4}P_{k-6} \cdots P_2\})$.

Proof. Again, direct multiplication gives verification.

Thus for k even the effect of complete subgeneration is to permute the k marks by a cycle. For examples of the effects given by these two lemmas, compare the arrangements at steps 1 and 24 and at steps 25 and 48 ($k = 4$) and at steps 1 and 120 ($k = 5$).

Consider now any such subgeneration beginning with the arrangement, say $(m_1, m_2, \cdots, m_k, \cdots m_n)$. With k even, each of the $k - 1$ applications of rule I finds a new mark to put in the kth position, since during this subgeneration t_k is assuming the values $1, 2, \cdots k$, and so by the special construction of rule II (and Lemma 1), position $k - 1$ contains successively, at the time of application of rule I, $m_{k-2}, m_{k-1}, m_{k-3}, m_{k-4}, \cdots, m_1$. With k odd, each of the $k - 1$ applications of rule II finds the $k - 1$ leftmost marks permuted by a $(k - 1)$-cycle (by Lemma 2), and hence also finds a new mark to put in the kth position. A simple induction now shows that such subgenerations yield $k!$ distinct arrangements. We have proved the following:

THEOREM. *The generation scheme as given above yields all* $n!$ *arrangements of* n *marks in exactly* $n! - 1$ *steps (starting from a given arrangement).*

4. Remarks. As in most other generation schemes the property of changing the jth mark only when all arrangements of the previous $j - 1$ marks have been generated allows significant time-savings in some problems. If following a transposition (P_iP_{k+1}) with $i < k + 1$ the problem decides it does not need to use the $k!$ arrangements formed by permuting the present k leftmost marks, then this subgeneration may be skipped by applying Lemma 1 or Lemma 2 according as k is odd or even. This immediately prepares the arrangement for the next application of (P_iP_{k+1}). The permutation of Lemma 2 is not a transposition, but is quite easy to code into the scheme. An interesting question is whether or not an equally simple generation by transposition scheme exists in which block skipping is also always done by transposition.

With the assumption that the marks being permuted are in most problems indices used for address modification, and thus should occupy the address portion of a computer word, a time comparison [2] between this scheme and the Tompkins-Paige method was made on Maniac II. With nine marks the transposition scheme generates arrangements about twenty per cent faster. In addition, the transposition

scheme is advantageous for problems in which minimum mixing of the marks at each step is important.

5. Acknowledgements. This work was performed under the auspices of the U. S. Atomic Energy Commission.

The need for such a scheme was suggested by Professor D. H. Lehmer of the University of California at Berkeley.

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Chebyshev Approximations to the Gamma Function

By Helmut Werner and Robert Collinge

In this note several Chebyshev approximations are given for the function $y = \Gamma(x + 2)$ for x in the $0 \leq x \leq 1.0$ range. The approximations were obtained from a table of $\Gamma(x + 2)$, employing well-known methods as described in numerous papers; see for instance [1] and the literature quoted there. The table of $\Gamma(x + 2)$ was calculated from the asymptotic expansion of $\log \Gamma(z)$ as given in [2] to provide data accurate to at least 10^{-21}. Compare also [3].

The asymptotic expansion of $\ln \Gamma(z)$ is given by

$$\ln \Gamma(z) = (z - \tfrac{1}{2}) \ln z - z + \ln \sqrt{2\pi} + \Phi(z)$$

where

$$\Phi(z) = \sum_{r=1}^{n} \frac{(-1)^{r-1}B_r}{2r(2r-1)} \frac{1}{z^{2r-1}} + R_n(z),$$

and B_r is the rth Bernoulli number.

It can be shown [2] that for $z > 0$ the value of $\Phi(z)$ always lies between the sum of n terms and the sum of $(n + 1)$ terms of the series, for all values of n. In terminating this series with the nth term the error $R_n(z)$ will be less than

$$\frac{B_{n+1}}{2(n+1)(2n+1)} \cdot \frac{1}{z^{2n+1}}.$$

By truncating $\Phi(z)$ at the 10th term it is easily shown that for values of $z \geq 13$, the error in the expansion is less than 5.5×10^{-22}. We therefore replace $\Phi(z)$ by $\sum_{i=1}^{10} A_i/z^{2i-1}$ and calculate $\ln \Gamma(z)$ for values of z in the range $13 \leq z \leq 14$.

Received July 25, 1960.

TABLE 1
Table of Coefficients

n	7	8	10
$\epsilon_{max}^{(n)}$	0.25×10^{-7}	0.16×10^{-9}	0.74×10^{-11}
$\nu = 0$	0.99999 99758	0.99999 99998 452	0.99999 99999 9269
1	0.42278 74605	0.42278 43662 730	0.42278 43369 6202
2	0.41177 41955	0.41183 92935 920	0.41184 02517 9616
3	0.08211 17404	0.08159 03449 474	0.08157 82187 8492
4	0.07211 01567	0.07416 00915 535	0.07423 79076 0629
5	0.00445 11400	0.00007 55964 181	−0.00021 09074 6731
6	0.00515 89951	0.01033 20685 065	0.01097 36958 4174
7	0.00160 63118	−0.00157 80074 635	−0.00246 67479 8054
8		0.00079 62464 760	0.00153 97681 0472
9			−0.00034 42342 0456
10			0.00006 77105 7117

n	13	15
$\epsilon_{max}^{(n)}$	0.96×10^{-14}	0.97×10^{-16}
$\nu = 0$	0.99999 99999 99990 44	0.99999 99999 99999 9032
1	0.42278 43351 02334 79	0.42278 43350 98518 1178
2	0.41184 03301 66781 29	0.41184 03304 21981 4831
3	0.08157 69261 24155 46	0.08157 69194 01388 6786
4	0.07424 89154 19444 74	0.07424 90079 43401 2692
5	−0.00026 61865 94953 06	−0.00026 69510 28755 5266
6	0.01114 97143 35778 93	0.01115 38196 71906 6992
7	−0.00283 64625 20372 82	−0.00285 15012 43034 6494
8	0.00206 10918 50225 54	0.00209 97590 35077 0629
9	−0.00083 75646 85135 17	−0.00090 83465 57420 0521
10	0.00037 53650 52263 07	0.00046 77678 11496 4956
11	−0.00012 14173 48706 32	−0.00020 64476 31915 9326
12	0.00002 79832 88993 83	0.00008 15530 49806 6373
13	−0.00000 30301 90810 28	−0.00002 48410 05384 8712
14		0.00000 51063 59207 2582
15		−0.00000 05113 26272 6698

n	17	18
$\epsilon_{max}^{(n)}$	0.10×10^{-17}	0.10×10^{-18}
$\nu = 0$	0.99999 99999 99999 99901 2	0.99999 99999 99999 99990 02
1	0.42278 43350 98467 79580 6	0.42278 43350 98467 21319 64
2	0.41184 03304 26367 20638 1	0.41184 03304 26430 62304 23
3	0.08157 69192 50260 90508 9	0.08157 69192 47528 84581 87
4	0.07424 90106 80090 41696 9	0.07424 90107 42094 91715 38
5	−0.00026 69810 33348 38176 8	−0.00026 69818 88740 38315 07
6	0.01115 40360 24034 39169 2	0.01115 40438 29069 91793 28
7	−0.00285 25821 44619 65607 6	−0.00285 26318 64702 11862 89
8	0.00210 36287 02459 83329 2	0.00210 38579 20672 20524 09
9	−0.00091 84843 69099 08014 2	−0.00091 92675 95039 95026 11
10	0.00048 74227 94476 75810 4	0.00048 94361 06998 14458 34
11	−0.00023 47204 01891 94985 9	−0.00023 86428 33752 63647 10
12	0.00011 15339 51966 59947 0	0.00011 73283 10224 09396 51
13	−0.00004 78747 98383 44672 4	−0.00005 43183 86280 13508 99
14	0.00001 75102 72717 90508 0	0.00002 28140 41153 66022 75
15	−0.00000 49203 75090 42313 2	−0.00000 80523 43363 48309 46
16	0.00000 09199 15640 71621 4	0.00000 21741 77495 45532 64
17	−0.00000 00839 94049 59039 7	−0.00000 03889 70057 38769 55
18		0.00000 00339 81801 01810 43

For the convenience of the reader the A_i coefficients are quoted below, to 25 significant figures.

$$
\begin{aligned}
A_1 &= 0.08333\ 33333\ 33333\ 33333\ 33333\ 3 \\
A_2 &= -0.00277\ 77777\ 77777\ 77777\ 77777\ 78 \\
A_3 &= 0.00079\ 36507\ 93650\ 79365\ 07936\ 508 \\
A_4 &= -0.00059\ 52380\ 95238\ 09523\ 80952\ 381 \\
A_5 &= 0.00084\ 17508\ 41750\ 84175\ 08417\ 508 \\
A_6 &= -0.00191\ 75269\ 17526\ 91752\ 69175\ 27 \\
A_7 &= 0.00641\ 02564\ 10256\ 41025\ 64102\ 56 \\
A_8 &= -0.02955\ 06535\ 94771\ 24183\ 00653\ 6 \\
A_9 &= 0.17964\ 43723\ 68830\ 57316\ 49385 \\
A_{10} &= -1.39243\ 22169\ 05901\ 11642\ 7432 \\
\ln\sqrt{2\pi} &= 0.91893\ 85332\ 04672\ 74178\ 03297
\end{aligned}
$$

A triple precision logarithm routine was used to evaluate $\ln z$, and then an exponential routine to calculate $\Gamma(z) = e^{\ln\Gamma(z)}$. Each of these routines produces results accurate to at least 24 significant digits.

After obtaining a table of $\Gamma(z)$ for z in the range $13 \leqq z \leqq 14$, we made use of the recursion formula $\Gamma(z + 1) = z\Gamma(z)$ in order to obtain a table of $\Gamma(x + 2)$ for x in the range $0 \leqq x \leqq 1.0$.

From the tests made on the results obtained, the values of $\Gamma(x + 2)$ were shown to be accurate to at least 21 significant figures.

Several Chebyshev approximations have been calculated to provide varying degrees of accuracy. Let

$$\Gamma(2 + x) = \sum_{r=0}^{n} a_r^{(n)} x^r + \epsilon_n(x)$$

and

$$\epsilon_{\max}^{(n)} = \max_{0 \leq x \leq 1} | \epsilon_n(x) |.$$

Table 1 gives the coefficients $a_r^{(n)}$ for $n = 7, 8, 10, 13, 15, 17, 18$ together with the corresponding $\epsilon_{\max}^{(n)}$.

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1. E. L. Stiefel, "Numerical methods of Tchebycheff approximation," *On Numerical Approximation* by R. E. Langer, The University of Wisconsin Press, Madison, 1959, p. 217–232.
2. E. T. Whittaker & G. N. Watson, *A Course of Modern Analysis*, Cambridge, 1958, p. 251–253.
3. M. E. Sherry & S. Fulda, "Calculation of gamma functions to high accuracy," *MTAC*, v. 13, 1959, p. 314–315.
4. H. S. Uhler, "Log π and other basic constants," *Proc. Nat. Acad. Sci. U. S. A.*, v. 24, 1938, p. 23–30.
5. H. S. Uhler, "The coefficients of Stirling's series for log $\Gamma(x)$," *Proc. Nat. Acad. Sci. U. S. A.*, v. 28, 1942, p. 59–62.

A Partition Test for Pseudo-Random Numbers

By J. C. Butcher.

A frequently used test on random digit sequences consists in forming from them sets of n integers; $\nu_1, \nu_2, \cdots, \nu_n$ say, which, on the hypothesis of randomness, are independent and have equal probabilities, $1/x$, of being each of the integers $1, 2, \cdots, x$ (x usually being chosen as a power of 2). From any equalities that may exist between $\nu_1, \nu_2, \cdots, \nu_n$ a partition of n is defined and the actual test is on the frequency of occurrence of the different partitions of n.

For example, a popular form of the test known as the "poker test" distinguishes the different partitions of $n = 5$ known by the descriptive names "all different," "one pair," "two pairs," "three of a kind," "full house" (three of one kind, two of another), "four of a kind" and "five of a kind."

It is obvious that much computing time must be absorbed in distinguishing between the various possibilities, and that if n is much greater than 5, the number of possible partitions becomes very large. For this reason it may be advantageous to group the partitions together in some way that lowers the number of cases and also simplifies the programming necessary to distinguish these cases.

A convenient way of doing this will now be described. It has the added advantage that the calculation of the expected probabilities is extremely simple. Let us distinguish n classes of partitions $C_1, C_2, \cdots, C_n$ where C_r includes all partitions defined from exactly r different integers occurring amongst $\nu_1, \nu_2, \cdots, \nu_n$. In the example $n = 5$, C_1 would include the single partition "five of a kind," C_2 would include the two partitions "full house" and "four of a kind," C_3 would include "three of a kind" and "two pairs," while C_4 and C_5 would each contain the single partitions "one pair" and "all different," respectively.

If the n integers $\nu_1, \nu_2, \cdots, \nu_n$ are generated one after the other, and if we define $S_i (i = 1, 2, \cdots, n)$ as the set of integers $1, 2, \cdots, x$, other than those identical with ν_j for some $j < i$, then the value of r that characterizes the class C_r is given by

$$r = \sum_{i=1}^{n} \epsilon_i ,$$

where ϵ_i is equal to 1 or 0 according as ν_i is or is not a member of S_i.

When x is no more than the computer word length, then S_i is conveniently represented by the binary digits of a single word, and this word is quickly computed from the word representing S_{i-1} if, for example, a logical conjunction instruction is available.

To find the expected frequency of occurrence of the class C_r for $r = 1, 2, \cdots, n$ we must find the probability p_r that the class C_r will arise in a single trial. It is easy to see that

$$p_1 = \frac{1}{x^{n-1}} a_1 ,$$

Received June 30, 1960.

$$p_2 = \frac{(x-1)}{x^{n-1}} a_2,$$

$$\vdots$$

$$p_n = \frac{(x-1)(x-2)\cdots(x-n+1)}{x^{n-1}} a_n,$$

where $a_1, a_2, \cdots, a_n$ are dependent on n, but not on x. Since $p_1 + p_2 + \cdots + p_n$ is identically equal to unity, we find

$$x^n = a_1 x + a_2 x(x-1) + \cdots + a_n x(x-1) \cdots (x-n+1) \tag{1}$$

for all values of x. The coefficients $a_1, a_2, \cdots, a_n$ are thus identical with Stirling's numbers of the second kind of order n [1], and they can be evaluated in turn by substituting $x = 1, 2, \cdots, n$ in (1) and using as a check the value $a_n = 1$.

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1. C. Jordan, *The Calculus of Finite Differences*, Chelsea Publishing Co., New York, 1947, p. 170.

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

34[D, E, L, S, X].—Nelson A. Logan, *General Research in Diffraction Theory*, Volumes I and II, Missiles and Space Division, Lockheed Aircraft Corp., Sunnyvale, California. Reports LMSD-288087 and LMSD-288088, December 1959, xxiii + 345 p. and xviii + 268 p., 28 cm. Deposited in UMT File.

Volume I consists of a study of the theory and applications of a class of integrals defined by

$$A_m^n(\xi) = \sum_{s=1}^{\infty} \alpha_s^n [Ai'(-\alpha_s)]^{-m} e^{-(\sqrt{3}-i)\alpha_s \xi/2},$$

$$B_m^n(\xi) = \sum_{s=1}^{\infty} \beta_s^{n-1} [Ai(-\beta_s)]^{-m} e^{-(\sqrt{3}-i)\beta_s \xi/2}$$

where α_s, β_s denote the roots defined by $Ai(-\alpha_s) = 0$, $Ai'(-\beta_s) = 0$, and $Ai'(-\alpha_s)$, $Ai(-\beta_s)$ are the turning values of the Airy integral. This representation is useful only for $\xi > 0$. Alternative representations useful for $\xi \to 0$ are developed for the case A_0^n and B_0^n. For $m = 1$ the functions are entire functions of ξ, and tables are given for the coefficients of the Taylor series of A_1^n and B_1^n. These coefficients are evaluated by using the Euler summation scheme to sum the divergent series obtained by setting $\xi = 0$, $m = 1$ in the above representations. When $m = 2$ it is necessary to extract some terms which are singular at $\xi = 0$. The remaining parts of A_2^n and B_2^n are shown to be entire functions. Tables for the coefficients in the Taylor series for these non-singular parts are found by using the Euler-Maclaurin summation formula to sum the divergent series which are obtained by setting $\xi = 0$, $m = 2$ in the above representations. For $\xi \to -\infty$, asymptotic expansions are obtained for the cases $m = 1$ and $m = 2$. Tables are given for the coefficients in these expansions.

Volume II consists of a set of 26 tables and 17 figures that provide a supplement to the theoretical analysis contained in Volume I. Tables A, B, and C contain special tables of exponential and trigonometric functions which will facilitate computation with residue series and asymptotic expansions of the diffraction integrals. The functions tabulated in the remaining tables can be used to study diffraction effects when (a) source and receiver are on the surface, (b) source (or receiver) is on the surface and the receiver (or source) is at a great distance, and (c) both source and receiver are at a great distance.

Author's Summary

35[F].—R. Kortum & G. McNiel, *A Table of Quadratic Residues for all Primes less than 2350*, LMSD 703111, October 1960, Lockheed Missiles and Space Division, Sunnyvale, California, iii + 3 + 378 unnumbered pages, 28 cm.

This large report, bound with a plastic spiral, lists all 187,255 quadratic residues of the 347 primes from 3 to 2347. The tables were computed on an IBM 7090 in about ten minutes. Presumably most of this time was spent in binary-decimal conversion and in writing on tape. The original printing was done on a high-speed, wire

matrix printer and is readable, but certainly not elegant. In compensation, the tables are very easy to use, since the spiral binding allows the pages to lie flat.

The tables also give $N(p)$, the number of positive non-residues $<\frac{1}{2}p$. In the introduction it is pointed out that for primes of the form $4m + 3$ we have

$$[\tfrac{1}{2}(p - 1)]! \equiv (-1)^{N(p)} \pmod{p}.$$

It is also indicated that for all such primes (but we must add >3) the class number, $h(-p)$, is given by

$$h(-p) = -\frac{1}{p}\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)a.$$

The much more easily computed formula [1],

$$h(-p) = \frac{p - 1 - 4N(p)}{4 - 2\,(2/p)},$$

is not mentioned. The introduction also states that it can be "found" in the table that $N(p) = m$ for all primes of the form $4m + 1$. But surely one does not need the table to be convinced of this simple theorem. The quantity which is really useful for those primes is $2\sum_{a=1}^{m}(a/p)$, and not the redundant $N(p)$.

D. S.

1. E. Landau, *Aus der elementaren Zahlentheorie*, Chelsea Publishing Co., New York, 1946, p. 128.

36[G X].—V. N. Faddeeva, *Computational Methods of Linear Algebra*, Translated by Curtis D. Benster, Dover Publications, Inc., New York, 1959, x + 252 p., 21 cm. Price $1.95.

The first chapter of this book forms a clear and well-written introduction to the elementary parts of linear algebra. The second chapter deals with numerical methods for the solution of systems of linear equations and the inversion of matrices, and the third with methods for computing characteristic roots and vectors of a matrix. Most of the important material in these domains is to be found here, and many numerical examples which illustrate the algorithms and point out their merits and deficiencies are given.

The discussion is directed principally to the hand computer, and machine computation in the modern sense is hardly present, but the book must be regarded as a valuable guide for the worker in the general area of linear computation.

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37[G, X].—M. Midhat J. Gazalé, *Les Structures de Commutation à m Valeurs et Les Calculatrices Numériques*, Collection de Logique Mathématique, Série A, Monographies Réunies par Mme. P. Fevrier, Gauthier-Villars, Paris, 1959, 78 p., 24 cm. Price 16 NF.

The theme of this pamphlet is sets of operations which are complete in the sense that "conjunction" and "negation" (or "exclusive or," "conjunction" and "1," or

the "Sheffer stroke") are complete. By an operation we mean a (single-valued) function whose domain is S^n for some set S and positive integer n. A set F of operations on S is complete if any operation f on S (of any number of arguments) can be constructed from F by composition (substitution) and identification of variables.

The first three chapters, which are introductory, include, among other things, a discussion of how constructing f from F corresponds to constructing a network "realizing" f from elements (primitive networks) which realize the elements of F. Chapter IV is preparatory to Chapter V, where the first significant theorem appears. This is to the effect that sum modulo p (p a prime), product modulo p, and the constant functions are complete. Alternatively, every n-ary operation on $0, 1, 2, \cdots, p - 1$ is representable by a polynomial in n variables over the field of integers, modulo p. The author fails to note, however, that for any finite field, any operation on the p^n field elements is representable by a polynomial over the field. As a matter of fact, essentially the same argument the author gives for p elements is applicable to the more general situation.

Chapter VI deals with a theorem of Webb to the effect that the binary operation W defined over $0, 1, \cdots, m - 1$ by $W(x, y) = 0$ if $x \neq y$ and by $W(x, y) = x + 1$, mod m, if $x = y$ is complete. The author gives a formulation of the theorem which does not make use of the additive structure on the set, and gives a proof of it.

The last chapter (VII) generalizes a theorem of E. L. Post to the effect that if R is a permutation of E, the integers mod m, then the pair of operations $\otimes_R$, P_R is complete, where $R(i) \otimes_R R(j) = R$ (min i, j) and $P_R(R(i)) = R(i + 1)$, for all $i, j \varepsilon E$.

The following misprints were discovered:

p. 39 line 10, $E = (0, 1 \cdots, n - 1)$ *should read* $E = (0, 1, \cdots, p - 1)$;
p. 40 (63), *read* A_{rq} *for* Arq;
p. 40 line 6 from bottom, *read* M_{rt} *for* Mrt;
p. 51 line 2 from bottom, *read* p^n terms *for* p terms;
p. 55, 56; each recurrence of bg *should read* b_g;
p. 59 last line, $a = \delta ab$ *should read* $q = \delta ab$.

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38[I].—A. O. GELFOND, *Differenzenrechnung*, Deutscher Verlag, Berlin, 1958, viii + 336 p., 23 cm. Price DM 40.

This is a translation of the Russian edition (1952) which is a revised and extended version of an earlier book (1936). It is mainly concerned with problems in the complex domain, and some material, traditional in courses on Finite Differences, is omitted. The techniques used are those of classical analysis. There are occasional sets of problems, and some very interesting worked examples.

The book is divided into three large chapters (1, 2, 5) and two smaller ones (3, 4). Chapter One deals with the problem of interpolation ("construct an (ap-

proximate) expression for a function, given its values at a discrete set of points"), and includes an account of the Chebyshev theory.

Chapter Two deals with the Newton Series, first for equally spaced nodes, then for more general cases. The chapter concludes with applications of interpolation-theoretic methods to number-theoretic problems, in particular, to a proof of the theorem that α and $\beta = e^{\alpha}$ cannot both be algebraic, except for $\alpha = 0$.

The early part of Chapter Five is concerned with conventional material, including, for example, Ostrowski's proof of Hölder's result that $\Gamma(z)$ does not satisfy an algebraic differential equation; the latter part is concerned with work of the author (1951) on linear differential equations of infinite order, with constant coefficients.

Chapter Three is concerned with earlier (1937) research of the author on the construction of (entire) functions given their values at a series of points a_n, $a_n \to \infty$, and with related problems, e.g., the uniqueness of such functions.

Chapter Four contains standard material on the Summation Problem and the theory of Bernoulli numbers and polynomials; it includes, e.g., a proper account of the Euler-MacLaurin Summation Formula.

The book is clearly and precisely written. It can be recommended as an excellent source for many of the basic theorems in numerical analysis, and is a very suitable complement to such books as Natanson [1], which is largely concerned with the real domain.

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1. I. P. Natanson, *Konstrucktive Funktionentheorie*, Akad. Verlag, Berlin, 1955 [*MTAC* Review 9, v. 13, 1959, p. 64–67.]

39[I, X].—J. Kuntzmann, *Méthodes Numériques*, Dunod, Paris, 1959, xvi + 252 p., 25 cm. Price NF 36.00.

The author (who is a professor of applied mathematics of the Faculty of Sciences at Grenoble) admits his concern over the lack of a suitable textbook in numerical mathematics written in French. Rather than translate a foreign (to him) work, he decides to write a new book.

For various reasons he decides to limit his book almost exclusively to interpolation. The usual interpolation formulas (Newton-Gregory, Stirling, Gauss, Bessel, Everett, and Lagrange) are included for equally-spaced abscissas and also for divided differences as appropriate.

For the most part, approximation by the standard sets of polynomials (Legendre, Chebyshev, etc.) is avoided, but Bernoulli polynomials and Bernoulli numbers are discussed.

More general formulas for which the given data might be either values of the function or values of certain derivatives are discussed. Numerical integration is avoided, but interpolation for functions of two or more variables as well as of a complex variable is included. The last two chapters deal with the theory of interpolation for linear sums of special functions (exponentials, trigonometric sums, etc.).

Since the book was written to fulfill a need in France, and since there is no cor-

responding need in the United States, the reviewer feels that the book will have limited appeal to American numerical analysts.

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40[K].—W. S. CONNOR & MARVIN ZELEN, "Fractional factorial experiment designs for factors at three levels," *Nat. Bur. Standards Appl. Math. Ser.*, No. 54, May 1959, v + 37 p., 26 cm. For sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. Price $.30.

This is a sequel to NBS Applied Mathematics Series, No. 48 [1] which contains plans for fractional factorial designs for factors at two levels. The present compilation lists fractional factorial designs for factors at three levels as follows: for 1/3 replications, 2 for 4 factors and 3 each for 5, 6, and 7 factors; for 1/9 replications, 3 each for 6, 7, and 8 factors; for 1/27 replications, 3 each for 7, 8, and 9 factors; for 1/81 replications, 3 each for 8 and 9 factors; and for 1/243 replications, 3 each for 9 and 10 factors. For the same replication and number of factors the designs differ by the size of the blocks into which the treatment combinations are arranged. No main effects are confounded with other main effects or with two-factor interactions. Measurable two-factor interactions when the design is used as completely randomized or when treatments are grouped into blocks are listed. In addition, interactions confounded with blocks are given. Text material discusses the plan of the designs, loss of information, and the analysis of this type of designs.

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1. NBS APPLIED MATHEMATICS SERIES, No. 48, *Fractional Factorial Experiment Designs for Factors at Two Levels*, U. S. Gov. Printing Office, Washington, D. C., 1957 (*MTAC* Review 7, v. 12, 1958, p. 66).

41[K].—EDWIN L. CROW & ROBERT S. GARDNER, "Confidence limits for the expectation of a Poisson variable," *Biometrika*, v. 46, 1959, p. 441–453.

For any m, the inequalities $\sum_{i=c_1}^{c_2} p_i(m) \geqq 1 - \epsilon$, $\sum_{i=c_1+1}^{c_2+1} p_i(m) < 1 - \epsilon$, $\sum_{i=a+1}^{a+c_2-c_1} p_i(m) < 1 - \epsilon$ for all a, where $p_i(m) = e^{-m}m^i/i!$, define $c_1(m)$ and $c_2(m)$ uniquely. Define $m_L(c)$ to be the smallest m for which $c_2(m) = c$, and $m_U(c)$ to be the greatest m for which $c_1(m) = c$.

Table 1, p. 448–453, gives m_L and m_U to 2D for $\epsilon = .2, .1, .05, .01, .001$, and $c = 0(1)300$. The table was computed to 3D on an unspecified electronic computer; when the computed third place was a 5, the 5 was retained in the printed table.

Table 2, p. 444, compares the present confidence limits with the system δ_1 of Pearson & Hartley [1] and the system δ_3 of Sterne [2]; table 3, p. 445, compares them with the approximate formulas of Hald [3]; table 4, p. 446, compares them with the mean randomized confidence intervals of Stevens [4].

Reprints may be purchased from the Biometrika Office, University College, London, W.C. 1, under the title "Tables of confidence limits for the expectation of a

Poisson variable." Price: Two Shillings and Sixpence. Order *New Statistical Tables*, No. 28.

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1. E. S. PEARSON & H. O. HARTLEY, *Biometrika Tables for Statisticians*, Cambridge Univ. Press, 1954, p. 204–205. [*MTAC*, v. 9, 1955, p. 205–211].
2. T. E. STERNE, "Some remarks on confidence or fiducial limits," *Biometrika*, v. 41, 1954, p. 275–278. [*MTAC*, v. 9, 1955, p. 216].
3. ANDERS HALD, *Statistical Theory with Engineering Applications*, John Wiley & Sons, Inc., New York, 1952, p. 718.
4. W. L. STEVENS, "Fiducial limits of the parameter of a discontinuous distribution," *Biometrika*, v. 37, 1950, p. 117–139.

—F. G. FOSTER & D. H. REES, *Tables of the Upper Percentage Points of the Generalized Beta Distribution*, New Statistical Tables Series No. 26. Reprinted from *Biometrika*, vols. 44 & 45. Obtainable from the Biometrika Office, University College, London, W.C. 1, 30 p., 27 cm. Price five shillings.

In sampling from a k-variate normal population, let A and B be independent estimates, based on ν_1 and ν_2 degrees of freedom, respectively, of the population variance-covariance matrix. Let $\theta_1 \leqq \cdots \leqq \theta_k$ be the roots of the determinantal equation $|\theta\nu_1 A + (\theta - 1)\nu_2 B| = 0$. Then the distribution of θ_k is given by

$$I_x(k; p, q) = K \int_0^x d\theta_k \int_0^{\theta_k} d\theta_{k-1} \cdots \int_0^{\theta_2} d\theta_1 \prod_{i=1}^{k} \theta_i^{p-1}(1 - \theta_i)^{q-1} \prod_{i>j} (\theta_i - \theta_j)$$

where $p = \frac{1}{2}(|\nu_2 - k| + 1)$, $q = \frac{1}{2}(\nu_1 - k + 1)$. $I_x(1; p, q)$ is simply the incomplete-beta-function ratio $I_x(p, q)$. Foster & Rees argue that the 'generalized beta distribution' is *a* (not *the*) natural generalization of the Beta distribution from univariate to multivariate analysis of variance; for other generalizations see [1], [2], [3], [4].

The tables under review constitute a compilation of tables previously published in three papers by Foster and Rees [5], [6], [7]. Tabulated therein to 4D are values of the root of the equation

$$I_x(k; p, q) = P$$

for $P = .8(.05)\ .95, .99$;

$$k = 2, p = \tfrac{1}{2}, 1(1)\ 10, q = 2\ (1)\ (20)\ (5)\ 50, 60, 80;$$
$$k = 3, 4, p = \tfrac{1}{2}(\tfrac{1}{2})\ 4, q = 1\ (1)\ 96.$$

Two- to four-point Lagrangian interpolation in p and q is recommended; no specific accuracy is guaranteed.

The computations for $k = 2$ were carried out on the N.R.D.C. Elliott 401 computer at Rothamsted; for $k = 3, 4$, on the DEUCE computer of the English Electric Company. Tables for $P = .95, .99$ and $k = 2(1)6$ have been given by Pillai [8], [9].

Two examples are given of the application of the tables to the analysis of dis-

persion of means, and one example is given of such application to the analysis of dispersion of regressions.

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Princeton, New Jersey

1. S. S. WILKS, "Certain generalizations in the analysis of variance," *Biometrika*, v. 24, 1932, p. 471–494.
2. E. S. PEARSON & S. S. WILKS, "Methods of statistical analysis appropriate for k samples of two variables," *Biometrika*, v. 25, 1933, p. 353–378.
3. STATISTICAL RESEARCH GROUP, COLUMBIA UNIVERSITY, *Selected Techniques of Statistical Analysis*, McGraw-Hill Book Co., New York, 1947, chap. 3, p. 111–184.
4. J. NEYMAN, Editor, *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, Univ. California Press, Berkeley & Los Angeles, 1951, p. 23–41.
5. F. G. FOSTER & D. H. REES, "Upper percentage points of the generalized beta distribution, I," *Biometrika*, v. 44, 1957, p. 237–247. [*MTAC*, Rev. 165, v. 12, 1958, p. 302]
6. F. G. FOSTER, "Upper percentage points of the generalized beta distribution, II," *Biometrika*, v. 44, 1957, p. 441–453 [[*MTAC*, Rev. 167, v. 12, 1958, p. 302]
7. F. G. FOSTER, "Upper percentage points of the generalized beta distribution, III," *Biometrika*, v. 45, 1958, p. 492–503. [*Math. Comp.*, Rev. 77, v. 14, 1960, p. 386]
8. K. C. S. PILLAI, *Concise Tables for Statisticians*, Statistical Center, Univ. of the Philippines, Manila, 1957.
9. K. C. S. PILLAI & C. G. BANTEGUI, "On the distribution of the largest of six roots of a matrix in multivariate analysis," *Biometrika*, v. 46, 1959, p. 237–240.

43[K].—TOSIO KITAGAWA & MICHIWO MITOME, *Tables for the Design of Factorial Experiments*, Dover Publications, Inc., New York, 1955 (printed in Japan; originally published by the Baifukan Company of Japan as part 3 of the work with the same title), vii + 253 p., 26 cm. Price $3.00.

These tables consist of the actual tables that appeared in the original 1953 publication in Japanese. An introduction to design principles and an explanation of the mathematical principles, parts 1 and 2 of the first publication, have been omitted. Readers are now referred to Kitagawa's *Lectures on the Design of Experiments* for this information and presumably for some help in the use of these tables.

The American publisher's jacket states that "this book contains tables for the design of factorial experiments and covers Latin squares and cubes, factorial design, fractional replication in factorial design, factorial designs with split-plot confounding, factorial designs confounded in quasi-Latin squares, lattice designs, balanced incomplete block designs, and Youden's squares." The table of contents gives more detail under each of the eight main headings just listed, except for the last two. For example, orthogonal squares and cubes are listed, the 2^n series of factorial arrangements goes up through 2^9, mixtures of factorials such as $a^n b^m$ mostly for $m = 1$ are listed, and the factorial replicates cover the 2^n for $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$ replicates plus the $\frac{1}{3}$ replicate for 3^n. Perhaps it should be noted that tables such as these are not really "for the design of experiments"; the function of the tables is to help select a layout or make easy the randomization of the layout after the design has been selected.

An examination of these tables shows that four Japanese pages have been cut out with scissors, and four English pages pasted in their place. The jacket further describes these tables as a "New revised edition. Explanatory notes." The author's preface does not describe what this reviewer would call a 'revised edition' and the explanatory notes consist of only one page. Since Kitagawa's *Lectures on Design of Experiments* may not be readily available to some users of these tables, other

references might have been cited, e.g., O. Kempthorne, *Design and Analysis of Experiments*, W. G. Cochran and G. M. Cox, *Experimental Designs*, and O. L. Davies, *Design and Analysis of Industrial Experiments*.

These tables are excellently and clearly printed. After one becomes acquainted with their structure and arrangement the tables should prove useful on many occasions to those persons engaged in the design of experiments in any field. One unique feature of these tables deserves notice. A complete listing of all 576 configurations of the 4x4 Latin square is given. Continuation of this procedure for larger squares would have produced a bulky volume. One wonders about the special utility of 4x4 Latin squares which merited this complete listing.

There are two comments that must be made about these tables. The first comment is a criticism on the failure to include a table of random numbers within the volume. This reviewer's first act in using these tables will be to insert a small table of random numbers in both the front and rear of the volume. A table of experimental designs cannot be used without a random number table. As a consultant, when I pick up 'my tables,' I want to be sure that both items are with me.

The second comment follows from the first. A preliminary section on randomization procedures and choice of specific layout for each design should have been included. If omitted, specific references to such instructions in the Fisher & Yates tables or in O. Kempthorne's book should have been given. In this reviewer's experience both minor and major errors in designs have occurred because of a lack of clear understanding of proper randomization procedures.

Finally, one may remark that these tables would have been much improved by the inclusion of some explanatory materials, and references for each design included. For statisticians, R. A. Fisher & F. Yates, *Statistical Tables for Biological, Agricultural and Medical Research*, Oliver & Boyd Ltd., Edinburgh (Fifth edition, 1957), and E. S. Pearson & H. O. Hartley, *Biometrika Tables for Statisticians*, Vol. I, Cambridge, published for the Biometrika Trustees of the University Press (2nd printing, 1956) have set a high standard in this respect. The continued rapid development in the field of experimental design makes it difficult to keep tables of this type up to date. It is hoped that a really revised edition will soon appear. Designs for response surface investigation and new fractional factorial arrangements need to be readily available.

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44[K].—INGRAM OLKIN, SUDHISH G. GHURYE, WASSILY HOEFFDING, WILLIAM G. MADOW, & HENRY B. MANN, Editors, *Contributions to Probability and Statistics, Essays in Honor of Harold Hotelling*, Stanford University Press, 1960, x + 517 p., 24 cm. Price $6.50.

This volume contains a collection of forty-two essays on probability and mathematical statistics in honor of Professor Harold Hotelling on his sixty-fifth birthday. The list of contributors, limited to those who have been closely associated with

Professor Hotelling, looks nevertheless like an up-to-date "Who's Who" in the subject field. This fact alone pays an appropriate tribute to his influence and leadership.

The first two essays, fitting to the occasion, deal with Hotelling the man, and as a leader and teacher in the field of mathematical statistics. The third one is a reprint of Hotelling's own excellent paper on "Teaching of Statistics," and the fourth one is a bibliography of his work. A total of ninety papers, not including reviews, were credited to him between 1925 and 1959—a truly impressive record of accomplishment.

The remaining thirty-eight research papers cover a wide spectrum of topics. There are seven papers on design and analysis of experiments, and about the same number in non-parametric statistics and also multivariate problems. Investigations into power, optimality, consistency, and robustness of tests, distribution theorems, and stochastic processes make up the bulk of the remaining papers. There is one paper on the inversion of partitioned matrices (Greenberg and Sarhan) and one on the numerical convergence of iterative processes (Moriguti).

Since a listing of titles and authors takes about two pages, a detailed review of this diversified volume is an impossible task within the space allotted. If one paper has to be singled out as truly outstanding among the thirty-eight, I believe most people would agree to the choice of John Tukey's "A Survey of Sampling from Contaminated Distributions," which investigates the robustness of efficiency of competitive estimators. In the paper the author considers two normal populations which have the same mean but whose standard deviations are in the ratio 3:1. One of the questions asked was: "What fraction of the wider normal population must be added to the narrower one in order for the mean deviation to be as good a large sample measure of scale as the standard deviation?" The answer, given two pages later, turns out to be a shockingly low .008. Tukey then suggests that "Problems of robustness of efficiency are probably as important as problems of robustness of validity, and, because of their relatively undeveloped stage, deserve even more attention from statisticians." No doubt this suggestion will be heeded.

A list of titles and authors follows. Texts which are accompanied by tables are marked with an asterisk. The tables in paper No. 20 are separately described in the review immediately following. All the other tables are of illustrative nature, with limited selections of entries, and will not be discussed here.

Part I. An Appreciation

1. Harold Hotelling—William G. Madow
2. Harold Hotelling—A Leader in Mathematical Statistics—Jerzy Neyman
3. The Teaching of Statistics—Harold Hotelling
4. Bibliography of Harold Hotelling

Part II: Contributions to Probability and Statistics

5. Some Remarks on the Design and Analysis of Factorial Experiments—R. L. Anderson
6. A Limitation of the Optimum Property of the Sequential Probability Ratio Test—T. W. Anderson and Milton Friedman
7. Decision Theory and the Choice of a Level of Significance for the t-Test—Kenneth J. Arrow

8. Simultaneous Comparison of the Optimum and Sign Tests of a Normal Mean—R. R. Bahadur
9. Some Stochastic Models in Ecology and Epidemiology—M. S. Bartlett
10. Random Orderings and Stochastic Theories of Responses—H. D. Block and J. Marschak
11. On a Method of Constructing Steiner's Triple Systems—R. C. Bose*
12. A Representation of Hotelling's T^2 and Anderson's Classification Statistic W in Terms of Simple Statistics—Albert H. Bowker
13. Euler Squares—Kenneth A. Bush
14. A Compromise Between Bias and Variance in the Use of Nonrepresentative Samples—Herman Chernoff
15. Construction of Fractional Factorial Designs of the Mixed $2^m\,3^n$ Series—W. S. Connor
16. Application of Boundary Theory to Sums of Independent Random Variables—J. L. Doob, J. L. Snell, and R. E. Williamson
17. Some k-Sample Rank-order Tests—Meyer Dwass
18. Characterization of Some Location and Scale Parameter Families of Distributions—S. G. Ghurye
19. Generalization of Some Results for Inversion of Partitioned Matrices—B. G. Greenberg and A. E. Sarhan
20. Selecting a Subset Containing the Best of Several Binomial Populations—Shanti S. Gupta and Milton Sobel*
21. Consistency of Maximum Likelihood Estimation of Discrete Distributions—J. Hannan
22. An Upper Bound for the Variance of Kendall's "Tau" and of Related Statistics—Wassily Hoeffding
23. On the Amount of Information Contained in a σ-Field—Gopinath Kallianpur
24. The Evergreen Correlation Coefficient—M. G. Kendall
25. Robust Tests for Equality of Variances—Howard Levene*
26. Intrablock and Interblock Estimates—Henry B. Mann and M. V. Menon
27. A Bivariate Chebyshev Inequality for Symmetric Convex Polygons—Albert W. Marshall and Ingram Olkin
28. Notes on the Numerical Convergence of Iterative Processes—Sigeiti Moriguti
29. Prediction in Future Samples—George E. Nicholson, Jr.*
30. Ranking in Triple Comparisons—R. N. Pendergrass and R. A. Bradley*
31. A Statistical Screening Problem—Herbert Robbins
32. On the Power of Some Rank-order Two-sample Tests—Joan Raup Rosenblatt
33. Some Non-parametric Analogs of "Normal" ANOVA, MANOVA, and of Studies in "Normal" Association—S. N. Roy and V. P. Bhapkar
34. Relations Between Certain Incomplete Block Designs—S. S. Shrikhande
35. Infinitesimal Renewal Processes—Walter L. Smith
36. Classification Procedures Based on Dichotomous Response Vectors—Herbert Solomon
37. Multiple Regression—Charles Stein
38. An Optimum Replicated Two-sample Test Using Ranks—Milton E. Terry*
39. A Survey of Sampling from Contaminated Distributions—John W. Tukey
40. Multidimensional Statistical Scatter—S. S. Wilks

41. Convergence of the Empiric Distribution Function on Half-Spaces—J. Wolfowitz
42. Analysis of Two-factor Classifications With Respect to Life Tests—M. Zelen.*

The five editors are to be congratulated for assembling and presenting this volume in an excellent manner.

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45[K].—Shanti S. Gupta & Milton Sobel, "Selecting a subset containing the best of several binomial populations," p. 224–248, *Contributions to Probability and Statistics, Essays in Honor of Harold Hotelling*, edited by Olkin et al., Stanford University Press, 1960. [See preceding review.]

Given k binomial populations with unknown probabilities of success $p_1, p_2, \cdots p_k$, a procedure R is studied by the authors which selects a subset that guarantees with preassigned probability P^* that, regardless of the true unknown parameter values, it will include the best population; i.e., the one with the highest parameter value. Procedure R for equal sample sizes is given as follows. Retain in the selected subset only those populations for which $x_i \geqq x_{max} - d$, where $d = d(n, k, P^*)$ is a non-negative integer, and x_i denotes number of successes based on n observations from the ith population. Table 2 gives the values of d for $k = 2(1)20, 20(5)50$; $n = 1(1)20, 20(5)50, 50(10)100, 100(25)200, 200(50)500$; $P^* = .75, .90, .95, .99$ (a trial and error procedure R is given for large, unequal sample sizes).

Table 3 gives the expected proportion of populations retained in the selected subset by procedure R (for the special case $p_1 = p_2 = \cdots = p_{k-1} = p, p_k = p + \delta$, $0 \leqq \delta \leqq 1, 0 \leqq p \leqq 1 - \delta$) for $n = 5(5)25$; $p^* = .75, .90, .95$; $\delta = .00, .10, .25, .50$; and $p + \delta = .50, .75, .95, 1.00$.

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46[K].—Maurice Henri Quenouille, "Tables of random observations from standard distributions," *Biometrika*, v. 46, 1959, p. 178–202. Egon Sharpe Pearson, "Note on Mr. Quenouille's Edgeworth Type A transformation," *Biometrika*, v. 46, 1959, p. 203–204.

Quenouille offers a random sample of 1000 each from the normal distribution and seven specified non-normal distributions. While a sample of 1000 is too small for much serious Monte Carlo work, the method of construction of the present tables, where the normal sample uniquely and monotonely determines the 7 non-normal samples, makes it suitable for pilot studies of the sensitivity of statistical procedures to departures from normality.

Specifically, let x_1 be a unit normal deviate from the tables of Wold [1]. Define

$$y = (2\pi)^{-1/2} \int_{-\infty}^{x_1} \exp\left(-\tfrac{1}{2}x^2\right) dx,$$
$$x_2 = 3^{1/2}[2y - 1],$$

$$
\begin{aligned}
x_3 &= 0.46271\, e^{x_1} - 0.76287,\\
x_4 &= -1 - \log_e (1 - y),\\
&\begin{cases} 124416\, x_5 = -9552 + 127225\, x_1 + 7824\, x_1^2 - 40\, x_1^3 + 576\, x_1^4 - 252\, x_1^5, & x_1 > -2.5,\\ x_5 = -1.86,\ x_1 \leqq -2.5, & \end{cases}\\
1536\, x_6 &= 1411\, x_1 + 56\, x_1^3 - 3\, x_1^5,\\
124416\, x_7 &= -12144 + 122878\, x_1 + 14304\, x_1^2 - 1066\, x_1^3 - 720\, x_1^4 + 261\, x_1^5,\\
&\begin{cases} x_8 = -2^{-1/2} \log_e [2 - 2y], & x_1 > 0\\ x_8 = 2^{1/2} \log_e 2y, & x_1 \leqq 0. \end{cases}
\end{aligned}
$$

Then, for $i = 1(1)8$, $E(x_i) = 0$, $E(x_i^2) = 1$. Here x_2 is a rectangular random variate; x_3, a log-normal variate; x_4, a one-tailed exponential variate; x_8, a two-tailed exponential variate; x_5, x_6, x_7 are Cornish-Fisher expansions with specified κ_3 and κ_4. A short table on p. 179 shows that the specifications are not met precisely; Pearson's note shows that this failure is negligible for samples of 1000.

The main table, p. 183–202, gives 1000 values of x_i, $i = 1(1)8$, to 2 D, with Σx and Σx^2 in blocks of 50. Auxiliary tables on p. 180–182 give the first and second sample moments of the x_i; their theoretical κ_3, κ_4, κ_5, κ_6; frequency distributions of the 8 samples; x_5, x_6, x_7 to 3 D for $x_1 = -3.2(.1) + 3.2$. The italic headlines on p. 181–182 should be interchanged.

It is not clear why random normal numbers were used as the basis for this table rather than random rectangular numbers, nor why the 2 D deviates of Wold [1] were chosen over the 3 D deviates of Rand Corp. [2].

Reprints may be purchased from the Biometrika Office, University College, London, W.C. 1, under the title "Tables of 1000 standardized random deviates from certain non-normal distributions." Price: Two Shillings and Sixpence. Order New Statistical Tables, No. 27.

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1. Herman A. O. Wold, *Random Normal Deviates*. Tracts for Computers, no. 25, Cambridge Univ. Press, 1954.
2. The RAND Corporation, *A Million Random Digits With 100,000 Normal Deviates*, The Free Press, Glencoe, Illinois, 1955. [*MTAC*, v. 10, 1956, p. 39–43].

47[K].—Alfred Weissberg & Glenn H. Beatty, *Tables of Tolerance-Limit Factors for Normal Distributions*, Battelle Memorial Institute, 1959, 42 p., 28 cm. Available from the Battelle Publications Office, 505 King Avenue, Columbus 1, Ohio.

The abstract of the booklet reads as follows: "Tables of factors for use in computing two-sided tolerance limits for the normal distribution are presented. In contrast to previous tabulations of the tolerance-limit factor K, we tabulate the factors $r(N, P)$ and $u(f, \gamma)$, whose product is equal to K. This results in greatly increased compactness and flexibility. The mathematical development is discussed, including methods used to compute the tabulated values and a study of the accuracy of the basic approximation. A number of possible applications are discussed and examples given."

Since the mean μ and the standard deviation σ are frequently unknown, the toler-

ance limits must be computed on the basis of a sample estimate $\bar{x}$ of the mean and an estimate s of the standard deviation. The tolerance limits treated in the booklet have the form $x \pm Ks$, where the factor K (the product of the tabulated entries $r(N, P)$ and $u(f, \gamma)$) accounts for sampling errors in $\bar{x}$ and s as well as for the population fraction P.

Six levels of probability for P and γ are used (.50, .75, .90, .95, .99, .999). The values of N used are given by

$$N = 1(1)300(10)1000(1000)10000, \infty.$$

The values of f used are given by

$$f = 1(1)1000(1000)10000, \infty.$$

Values of $r(N, P)$ and $u(f, \gamma)$ are given to four decimal places, which means that most of the tabular entries have five significant figures.

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48[K].—E. J. Williams, *Regression Analysis*, John Wiley & Sons, Inc., New York, 1959, ix + 214 p., 24 cm. Price $7.50.

This useful volume is a monograph devoted to the exposition of the practical aspects of "regression analysis." These so-called regression analysis techniques are based on the method of least squares and are equivalent to analysis of variance procedures. The author discusses many different techniques, some containing much novelty. All are accompanied by illustrations using actual data drawn mainly from the biological sciences. The book contains a great deal of interesting discussion and advice on the proper and practical applications of the methods.

No attention is devoted to the planning of experiments; the book is only concerned with the analysis of data. Although nearly all the techniques involve the solution of simultaneous equations, there is little discussion of numerical techniques, except to recommend the "Crout" method.

The author makes much use of statements about parameters which are termed fiducial statements. This reviewer feels these are confidence statements. In explaining the meaning of fiducial statements the author writes (p. 91), ". . . a fiducial statement about a parameter is, broadly speaking, a statement that the parameter lies in a certain range or takes a certain set of values. The statement is either true or false in any particular instance, but it is made according to a rule which ensures that such statements, *when applied in repeated sampling*, have a given probability (say 0.95 or 0.99) of being correct."

The various techniques are presented without theory, as "to have done so would have made the book unnecessarily long." Without the accompanying theoretical material, this book is simply a handbook of regression methods. It is for this reason

that the reviewer feels the book will be more useful to applied statisticians than to the author's intended audience, i.e., research workers in the experimental sciences.

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49[L].—G. F. MILLER, *Tables of Generalized Exponential Integrals*, National Physical Laboratory Mathematical Tables, Vol. 3, British Information Services, New York, 1960, iii + 43 p., 28 cm. Price $1.43 postpaid.

According to the author, the tables under review were prepared to meet the requirements of quantum chemists concerned with the evaluation of molecular integrals, who frequently have found the tables computed by the New York Mathematical Tables Project and edited by G. Placzek [1] inadequate for this purpose.

Actually tabulated in the present work is the auxiliary function

$$F_n(x) = (x + n)e^x E_n(x),$$

where $E_n(x)$ represents the generalized exponential integral, defined by the equation $E_n(x) = \int_1^\infty e^{-xt} t^{-n}\, dt$. Three tables of $F_n(x)$ to 8D are provided. The first table covers the range $x = 0(0.01)1$ for $n = 1(1)8$; the second, the range $x = 0(0.1)20$ for $n = 1(1)24$; and the last, the range $1/x = 0(0.001)0.05$ for $n = 1(1)24$. Modified second (and occasionally fourth) central differences are provided throughout for use with Everett's interpolation formula. For details of methods of interpolation and tables of interpolation coefficients the table-user is referred to the first two volumes of this series of tables [2], [3].

It is stated that the total error in an unrounded interpolated value of $F_n(x)$ derived from the present tables need never exceed $1\frac{1}{2}$ units in the eighth decimal place if the tabulated differences are used. Furthermore, the values of $F_n(x)$ here tabulated are guaranteed to be accurate to within 0.6 unit in the last place.

The tables are preceded by an Introduction containing a brief account of pertinent literature, followed by a section devoted to a description of the tables and a justification for the tabulation of $F_n(x)$ in preference to $E_n(x)$. The properties of the generalized exponential integral, many of them reproduced from Placzek [1], are enumerated in a third section. The fourth section of the text is devoted to a careful description of the several procedures followed in the preparation of the tables. An excellent set of references is appended to this introductory textual material.

The typography is uniformly excellent, and the format of the tables is conducive to their easy use. The only defect observed was a systematic error in the heading of Table 2 on pages 24 through 37, where this heading erroneously appears as Table 3.

J. W. W.

1. NATIONAL RESEARCH COUNCIL OF CANADA, Division of Atomic Energy, Report MT-1, *The Functions* $E_n(x) = \int_1^\infty e^{-xu} u^{-n}\, du$, Chalk River, Ontario, December 1946. Reproduced in

Nat. Bur. Standards Appl. Math. Ser. No. 37, *Tables of Functions and of Zeros of Functions*, 1954, p. 57–111. See RMT **392**, *MTAC*, v. 2, 1946–47, p. 272; and RMT **104**, *MTAC*, v. 10, 1956, p. 249–250.

2. L. Fox, *The Use and Construction of Mathematical Tables*, National Physical Laboratory Mathematical Tables, v. 1, London, 1956. See RMT **8**, *MTAC*, v. 13, 1959, p. 61–64.

3. L. Fox, *Tables of Everett Interpolation Coefficients*, National Physical Laboratory Mathematical Tables, v. 2, London, 1958.

50[L].—F. W. J. Olver, Editor, *Bessel Functions*, Part III, *Zeros and Associated Values*, Royal Society Mathematical Tables No. 7, Cambridge University Press, New York, 1960, lx + 79 p., 29 cm. Price \$9.50.

The present volume is a step towards the completion of a program for the tabulation of Bessel functions initiated by the British Association Mathematical Tables Committee, and continued since 1948 by the Royal Society Mathematical Tables Committee. Part I of this series, *Bessel Functions, Functions of Order Zero and Unity* appeared in 1937, and Part II, *Bessel Functions, Functions of Positive Integer Order* appeared in 1952 (see *MTAC* v. 7, 1953, p. 97–98). Recall that Part I contains a section on the zeros of $J_n(z)$, $Y_n(z)$, $n = 0, 1$, but Part II is without a section devoted to zeros.

Part III, the present work, deals with the evaluation of zeros of the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ for general ν and z. Tables are also provided as described later in this review. A history of the project is given in the "Introduction and Acknowledgements," by C. W. Jones and F. W. J. Olver. A chapter on "Definitions, Formulae and Methods" by the above authors is a valuable compendium of techniques for the enumeration of zeros and associated functions. In particular, it is an excellent guide if zeros are required of other transcendental functions which satisfy second-order linear differential equations. Several methods of computation are outlined. For instance, the method of McMahon is useful for ν fixed and z large, while the inverse interpolation approach of Miller and Jones presupposes a tabulation of the functions themselves. Between the regions covered by these techniques is a gap which increases with increasing ν. The gap is bridged by application of Olver's important contributions on uniform asymptotic expansions of Bessel functions.

The section "Description of the Tables, Their Use and Preparation" is by the editor. A short description of the tables follows. Table I gives zeros $j_{n,s}$ of $J_n(x)$, $y_{n,s}$ of $Y_n(x)$, and the values of $J_n'(j_{n,s})$, $Y_n'(y_{n,s})$. Table II gives zeros $j'_{n,s}$ of $J_n'(x)$, $y'_{n,s}$ of $Y_n'(x)$, and the values of $J_n(j'_{n,s})$, $Y_n(y'_{n,s})$. Table III gives zeros $a'_{m,s}$, $b'_{m,s}$ of the derivatives $j_m'(x)$, $y_m'(x)$ of the spherical Bessel functions $j_m(x) = (\pi/2x)^{1/2}J_{m+1/2}(x)$, $y_m(x) = (\pi/2x)^{1/2}Y_{m+1/2}(x)$, and the values of $j_m(a'_{m,s})$, $y_m(b'_{m,s})$. The ranges covered are

$$n = 0(\tfrac{1}{2})20\tfrac{1}{2}, \qquad s = 1(1)50, \quad \text{Tables I and II;}$$

$$m = 0(1)20, \qquad s = 1(1)50, \quad \text{Table III.}$$

All entries are to eight decimals, and in no case should the end-figure error exceed 0.55 of a unit in the eighth decimal.

The coefficients in the uniform asymptotic expansions (previously mentioned) which are used to evaluate items in Tables I–III for n (or m) large are given in Table IV. The expansions for the Bessel functions of Tables I–III also depend on zeros and associated values of certain Airy functions and their derivatives. These

data are recorded in Table V. Further description of the contents of Tables IV–V requires much more space and so is omitted here. Suffice it to say that with the aid of these tables, the entries in Tables I–III can with a few exceptions be evaluated to at least eight significant figures for $20 \leqq n < \infty$, $1 \leqq s < \infty$.

There is a good set of references. The printing and typography are excellent, and the present volume upholds the eminent tradition of British table-makers.

Y. L. L.

51[L, V].—J. W. Miles, "The hydrodynamic stability of a thin film of liquid in uniform shearing motion," *J. Fluid Mech.* 8, Pt. 4, 1960, p. 593–610. (Tables were computed by David Giedt.)

Let

$$\mathfrak{F}(z) = [1 - F(z)]^{-1} = w\,[A_i'(-w)]^{-1}\left[\frac{1}{3} + \int_0^\infty A_i(-t)\,dt\right], \qquad w = ze^{i\pi/6}.$$

$$\mathfrak{F}'(z) = z^{-1}\mathfrak{F}(z) + we^{i\pi/6}[A_i'(-w)]^{-1}A_i(-w)[1 - \mathfrak{F}(z)].$$

$$\mathfrak{F}^{(k)}(z) = \mathfrak{F}_r^{(k)}(z) + i\mathfrak{F}_i^{(k)}(z), \qquad k = 0, 1; \qquad F(z) = Fr(z) + iF_i(z).$$

The paper contains tables of $\mathfrak{F}(z)$, $\mathfrak{F}'(z)$, $F(z)$ and $z^3F_i(z)$ for $z = -6(.1)10$, 4S. The tables were obtained on an automatic computer by numerical integration of an appropriate differential equation. It can be seen from the above that the tables depend on values of the Airy integral $A_i(z)$, its derivative and integral along the rays $\pi/6$ and $-5\pi/6$ in the complex plane. Tables of $A_i(z)$ and its derivative are now available for complex z in rectangular form, but not in polar form. Also, tables of $\int_0^z A_i(\pm t)\,dt$ are available for z real. Thus, the given tables depend on values of some basic functions which, if available, would cut new ground. Unfortunately, the basic items were swallowed up in the automatic computation of $F(z)$. We have here a poor example of table making,—a practice which should not be emulated.

Y. L. L.

52[P].—Helmut Hotes (Compiler), *Wasserdampftafel der Allgemeinen Elektricitäts-Gesellschaft*, R. Oldenbourg, Munich, 1960, 48 p., 30 cm. DM 16 (Paperback).

There are two tables in this collection. Table I is a four-place table giving the temperature, the specific volume, the specific enthalpy, and the specific entropy as functions of the absolute pressure p. The last three dependent variables are given both for the fluid state and the gaseous state. The variable p ranges from 0.010 to 225,650 atmospheres, and the interval varies from 0.001 to 2000. Table II gives the specific volume, specific enthalpy and specific entropy as functions of temperature for constant pressure. Here p has the values 1, 5, 10 (10) to 400 atmospheres, and t varies from 0 (10) to 330 degrees centigrade.

The tables were calculated by expressing each of the dependent variables as polynomials in the pressure with coefficients as functions of the temperature or in some cases functions of the temperature and pressure. The error bounds given by

such approximations to the specific volume and specific enthalpy as functions of pressure and temperature are included in the collection.

A. H. T.

53[P, X, Z].—WARD C. SANGREN, *Digital Computers and Nuclear Reactor Calculations*, John Wiley & Sons, New York, 1960, xi + 208 p., 24 cm. Price $8.50.

As the author states in his preface, the primary objective of this book is to present to nuclear engineers and scientists an introduction to high speed reactor calculations. Since the appearance of the basic reference, *The Elements of Nuclear Reactor Theory* by Glasstone and Edlund, Van Nostrand, 1952, the entire complexion of actual reactor design calculations has changed as a result of the growth in speed and size of computing machines, and reactor design calculations represent today a significant part of scientific computing time on modern computers.

The outline of the book by chapters is

Chapter 1. Introduction
Chapter 2. Digital Computers
Chapter 3. Programming
Chapter 4. Numerical Analysis
Chapter 5. A Code for Fission-Product Poisoning
Chapter 6. Diffusion and Age-Diffusion Calculations
Chapter 7. Transport Equation—Monte Carlo
Chapter 8. Additional Reactor Calculations

In Chapter 1, the author reviews the tremendous parallel growth of high speed computing machines and nuclear reactors, and their present interplay. In Chapter 2, an introduction and description of present day computers is given. In Chapter 3, programming for computers is introduced. After some preliminary remarks (no proofs) about interpolation, numerical integration, matrices, etc., items which can be found in many well-known texts on elementary numerical analysis, the author treats in Chapter 4 the more relevant problem of the numerical approximation of partial differential equations by difference equations, and their solution by means of iterative methods. Also, the treatment of interface conditions, which arise naturally in heterogeneous reactors, is given.

In Chapter 5, a simple code for fission-product poisoning is followed from the physical and mathematical definitions through to the construction of a program in the Bell (Wolontis) system.

In Chapter 6, the longest chapter, the author describes diffusion calculations, extending from steady-state criticality problems for reactors to the solution of two- and three-dimensional multigroup diffusion equations. In Chapter 7, the S_n method of Carlson is described, along with the use of Monte Carlo methods for solving problems such as those encountered in shielding calculations.

In his primary aim, the author does succeed. Nevertheless, the reviewer, being quite familiar with this area, was most critical with respect to the age of the references, as most of the technical papers referred to had appeared prior to 1957. As no serious attempt was made to fill the gap between these earlier developments and the developments which have taken place in the reactor field in the last few years, many statements in the book are either somewhat obsolete or misleading. For example, the numerical inversion of tridiagonal matrix equations on page 74 by an

algorithm is not stated to be simply Gauss elimination applied to the matrix problem, and in fact the author states that this "method" has not appeared in textbooks as yet. The iterative methods of Young-Frankel, and Peaceman-Rachford are each discussed twice, (p. 84 and p. 144) and not one of the four definitions is completely accurate. The book is, however, the only existing bridge between *The Elements of Nuclear Reactor Theory* and present computational technique in the reactor field.

R. S. V.

54[S, W].—Ronald A. Howard, *Dynamic Programming and Markov Processes*, Technology Press & Wiley, New York, viii + 136 p., 23 cm. Price $5.75.

Consider a physical system S represented at any time t by a state vector $x(t)$. The classical description of the unfolding of the system over time uses an equation of the form $x(t) = F(x(s), s \leq t)$, where F is a prescribed operation upon the function $x(s)$ for $s \leq t$. In certain simple cases, this reduces to the usual vector differential equation $dx/dt = g(x)$, $x(0) = c$.

For a variety of reasons, it is sometimes preferable to renounce a deterministic description and to introduce stochastic variables. If we take $x(t)$ to be a vector whose i-th component is now the probability that the system is in state i at time t, and allow only discrete values of time, we can in many cases describe the behavior of the system over time quite simply by means of the equation $x(t + 1) = Ax(t)$. Here $A = (a_{ij})$, $i, j = 1, 2, \cdots, N$, is a transition matrix whose element a_{ij} is the probability that a system in state j at time t will be found in state i at time $t + 1$. Processes of this type are called Markov processes and are fundamental in modern mathematical physics.

So far we have assumed that the observer plays no role in the process. Let us now assume that in some fashion or other the observer has the power to choose the transition matrix A at each stage of the process. We call a process of this type a *Markovian decision process*. It is a special, and quite important, type of dynamic programming process; cf. Chapter XI of R. Bellman, *Dynamic Programming*, Princeton University Press, 1957.

Let us suppose that at any stage of the process, we have a choice of one of a set of matrices, $A(q) = (a_{ij}(q))$. Associated with each choice of q and initial state i is an expected single-stage return $b_i(q)$. We wish to determine a sequence of choices which will maximize the expected return from n stages of the process. Denoting the maximum expected return from an n-stage process by $f_i(n)$, the principle of optimality yields the functional equation

$$f_i(n) = \max_q \left[b_i(q) + \sum_{j=1}^{N} a_{ij}(q) f_j(n-1)\right].$$

In this form, the determination of optimal policies and the maximum returns is easily accomplished by means of digital computers; see, for example S. Dreyfus, *J. Oper. Soc. of Great Britain*, 1958. Problems leading to similar equations, resolved in similar fashion, arise in the study of equipment replacement and in continuous form in the "optimal inventory" problem; see Chapter Five of the book mentioned above and K. D. Arrow, S. Karlin, and H. Scarf, *Studies in the Mathematical Theory of Inventory and Production*, Stanford University Press, 1959.

As in the case of the ordinary Markov process, a question of great significance is that of determining the asymptotic behavior as $n \to \infty$. It is reasonable to suspect, from the nature of the underlying decision process, that a certain steady-state behavior exists as $n \to \infty$. This can be established in a number of cases.

The author does not discuss these matters at all. This is unfortunate, since there is little value to steady-state analysis unless one shows that the dynamic process asymptotically approaches the steady-state process as the length of the processes increases. Furthermore, it is essential to indicate the rate of approach.

The author sets himself the task of determining steady-state policies under the assumption of their existence. Granted the existence of a "steady state," the functions $f_i(n)$ have the asymptotic form $nc + b_i + o(1)$ as $n \to \infty$, where c is independent of i. The recurrence relations then yield a system of equations for c and the b_i.

This system can be studied by means of linear programming as a number of authors have realized; see, for example, A. S. Manne, "Linear programming and sequential decisions," *Management Science*, vol. 6, 1960, p. 259–268.

Howard uses a different technique based upon the method of successive approximations, in this case an approximation in policy space. It is a very effective technique, as the author shows, by means of a number of interesting examples drawn from questions of the routing of taxicabs, the auto replacement problem, and the managing of a baseball team.

The book is well-written and attractively printed. It is heartily recommended for anyone interested in the fields of operations research, mathematical economics, or in the mathematical theory of Markov processes.

RICHARD BELLMAN

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(Reprinted from Quarterly of Applied Mathematics)

55[V, X].—GERTRUDE BLANCH, KARL GOTTFRIED GUDERLEY & EMMA MARIAN VALENTINE, *Tables Related to Axial Symmetric Transonic Flow Patterns*, WADC Technical Report 59–710, 1960, Office of Technical Services, U. S. Department of Commerce, Washington 25, D. C., xlviii + 108 p., 27 cm.

The equations of motion of a compressible fluid are non-linear and are generally difficult to handle. In certain cases, such as in the flow past slender bodies of revolution, the equations can be approximated by much simpler ones. For subsonic and supersonic flow these approximating equations are linear. When the flow velocity is nearly equal to one, the approximate equation for the disturbance potential takes the non-linear form

$$-\Phi_x \Phi_{xx} + \Phi_{yy} + \frac{\Phi_y}{y} + \frac{1}{y^2} \Phi_{\omega\omega} = 0$$

when x, y, ω are cylindrical coordinates. K. G. Guderley and his colleague H. Yoshihara have studied the flow past slender bodies at Mach numbers close to one in a series of papers and in a book by Guderley, *Theorie schallnaher Strömungen*, Springer-Verlag, 1957.

The basic technique applied to axially symmetric flows is to find a basic solution Φ^B of the form

$$\Phi^B = y^{3n-2} f(\zeta)$$

where $\zeta = x/y^n$ and n is a constant. The variable f then satisfies the ordinary differential equation

$$(f' - n^2\zeta^2)f'' + (5n^2 - 4n)\zeta f' - (3n - 2)^2 f = 0.$$

Further solutions necessary to satisfy particular boundary conditions are then found by perturbing the basic solution by the function $\bar{\Phi}$, i.e.,

$$\Phi = \Phi^B + \bar{\Phi}(x, y, \omega).$$

Then $\bar{\Phi}$ is assumed to satisfy the linear equation

$$-\Phi_x^B \bar{\Phi}_{xx} - \Phi_{xx}^B \bar{\Phi}_x + \bar{\Phi}_{yy} + \frac{\bar{\Phi}_y}{y} + \frac{\bar{\Phi}_{\omega\omega}}{y^2} = 0.$$

Particular solutions of the equation are then found in the form

$$\bar{\Phi} = y^\nu g(\zeta) \cos m\omega; \qquad m = 0, 1, 2, \cdots$$

Then $g(\zeta)$ satisfies the ordinary differential equation

$$(f' - n^2\zeta^2)g'' + [f'' + (2\nu n - n^2)\zeta]g' + (m^2 - \nu^2)g = 0.$$

The solution of this equation leads to an eigenvalue problem with the eigenvalue ν.

The present report tabulates the functions f and g together with their derivatives and some other related functions. In Table 1 appear 6D values of $f(\zeta)$ and $df/d\zeta$ for $\zeta = -7.5(.1) - 3(.02)1$; in Table 2 similar information appears for $g(\zeta)$ and $dg/d\zeta$. These tabular data are given for several values of the eigenvalue ν.

A rather complete discussion of the mathematical problems involved is given in the introduction. The eigenvalues are found using a contour integration technique. It is stated that the numerical calculations are performed on an ERA 1103, with considerable pains taken to insure accuracy. The entries are stated to be correct to within one unit in the last place.

The tables should be quite useful to anyone interested in the study of special cases of transonic flow.

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56[W].—H. L. TOOTHMAN, *A Table of Probability Distributions useful in War Games and other Competitive Situations*, NRL Report 5480, U. S. Naval Research Laboratory, Washington, D. C., May 16, 1960, i + 91 p., 27 cm.

A player makes a maximum of $(2r - 1)$ plays. On odd-numbered plays he scores 1 with probability p_1 and 0 with probability $1 - p_1$; on even-numbered plays he is eliminated from subsequent play with probability p_2. The probability that he will score exactly n is

$$S_n = \binom{r}{n} p_1^n (1 - p_1)^{r-n} (1 - p_2)^{r-1} + \sum_{k=n}^{r-1} \binom{k}{n} p_1^k (1 - p_1)^{k-n} p_2 (1 - p_2)^{k-1}.$$

The probability, U_t, that m independent players score a total of exactly t is the coefficient of x^t in $(\Sigma S_n x^n)^m$.

The table on p. 6–91 gives U_t to 4D for $m = 1(1)4, r = 1(1)4, t = 0(1)mr$. For $p_1 = .01(.01).06(.02).22, .25, p_2 = 0(.05).2(.1).9$; for $p_1 = .3(.05).95, p_2 = 0(.01).02(.02).12, .15(.05).9$. U_t was computed to 9D or better on the NAREC, and each value was rounded to 4D individually; i.e., the U_t were not forced to sum to 1. Quadratic interpolation in p_1 or p_2 is stated to yield a maximum error of .0016.

The typography (photo-offset reproduction of Flexowriter script) is adequate but undistinguished; all decimal points are omitted from the body of the table.

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57[W, X].—S. VAJDA, *An Introduction to Linear Programming and the Theory of Games*, John Wiley & Sons, Inc., New York, 1960, 76 p., 22 cm. Price $2.25.

This book introduces the basic mathematical ideas of linear programming and game theory (mostly matrix games) in a form suitable for anyone who has had a little analytic geometry (and is not frightened by subscripts and double subscripts). Part I, on linear programming, begins with two examples, the second of which is a transportation problem, and then describes the simplex method of solving the transportation problem. Then comes the graphical representation of the general linear programming problem, followed by the general simplex method and a discussion of such complications as finding a first feasible solution, multiple solutions, and degeneracy. The chapter closes with the duality theorem.

Part II, on games, begins with two examples of matrix games, the second of which admits no saddle point, and introduces the concepts of mixed strategy and value. This is followed by a discussion of games in extensive form, and their normalization. A section on graphical representation is followed by the description of the equivalent linear program, and the Shapley-Snow "algorithm" is offered as an alternative method of calculating equilibrium strategies and value. Next the concept of equilibrium point in non-zero sum games is discussed, followed by three examples of infinite games. The book closes with an appendix proving the main theorem of matrix games along Ville's lines.

This book, compiled from lecture notes of short courses offered by the author, is suitable as a text for a short course for students with slight mathematical preparation.

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58[W, Z].—FRANZ L. ALT, Editor, *Advances in Computers*, Vol. 1, Academic Press, Inc., New York, 1960, x + 316 p., 24 cm. Price $10.00.

Advances in Computers is a useful addition to the rapidly growing literature on modern high-speed computers and their application. It is intended by the editor to be the first volume in a series which will contain monographs by specialists in vari-

ous areas of work in this field. These are to be written in non-technical language, so as to be easily understood by specialists in areas other than those of the writer.

The present volume contains six articles by well-qualified authors in six significant and interesting areas of work related to computers. Four summarize progress to date in the application of computers to weather prediction, translation of languages, playing games, and recognition of spoken words. Two are related to techniques used in computer programming and design. The titles and authors are:

1. General-Purpose Programming for Business Application—Calvin E. Gotlieb
2. Numerical Weather Prediction—Norman A. Phillips
3. The Present Status of Automatic Translation of Languages—Yehoshua Bar-Hillel
4. Programming Computers to Play Games—Arthur L. Samuel
5. Machine Recognition of Spoken Words—Richard Fatehchand
6. Binary Arithmetic—George W. Reitwiesner.

Since most of the areas of work covered by the papers in this volume are in a rapid state of flux, the assignment to write survey papers in these areas, undertaken by the authors, is a most difficult one. Each author has proceeded to carry out this assignment in his own characteristic manner. Thus, Gotlieb attempts to present a factual summary of some of the programming procedures used at present in processing data for business applications; whereas, Yehoshua Bar-Hillel presents a critical evaluation of the various efforts conducted in the field of automatic translation of languages—at times, highly critical. A large part of the material covered is admittedly subjective, and bears the imprint of the writers' points of view and contributions. Nevertheless, the six papers in this volume constitute authoritative surveys of the areas of work discussed. Together with the bibliographies given at the end of each paper, these articles will be valuable to the new researcher in the fields covered, as well as to the interested layman who wishes to familiarize himself with the exciting advances in computer technology.

H. P.

59[Z].—ANDREW D. BOOTH, *Automation and Computing*, The Macmillan Co., New York, 1959, 158 p., 21 cm. Price $5.00.

This book is intended mainly for the educated layman. In it the author attempts "to bridge the gap between the superficial accounts of electronic computers and automation . . . and the specialists' monographs. . . ." He has given an admirably written and lucid account of digital and analogue computers. His three chapters on the logical design of digital computers, the physical basis of this design, and programming for digital computers are very clear and informative, though concise.

The three chapters on automation in clerical work, control of continuous processes, and automatic machine tools and assembly processes are not as well done as the first three. The well-educated layman will have to expend a great deal of effort in order to follow the discussion in these chapters.

The last two chapters entitled "Strategic and Economic Planning" and "Non-numerical Applications of Computing Machines" are very brief. The former is much too short to give the reader more than a glimmer of what is involved in game theory. The last chapter furnishes a well-written introduction to methods for non-

numerical applications of computers, but, because it is so short, leaves the reader wishing the author had devoted more space to this subject. This reader would have preferred to have the author do this and omit some of his pronouncements on government (for example, the discussion on page 20 beginning with "Democratic government, too, is an example of Man in decay, . . .").

There are a few typographical errors in the book. The most disturbing one appears on page 36 where the binary addition table has the entry

$$1 + 1 = 1 \text{ (carry 1)}.$$

A. H. T.

60[Z].—ROBERT H. GREGORY & RICHARD L. VAN HORN, *Automatic Data-Processing Systems*, Wadsworth Publishing Co., San Francisco, 1960, xii + 705 p., 23 cm. Price $11.65.

This introductory book on automatic data-processing systems (ADPS) is a revision of a text which was used in management development courses sponsored by the Army Ordnance Corps. The affirmative objective is to instruct, enlighten, and inform management on the developments, techniques and applications of methods in management science, mathematics, and large-scale computing for the solution of today's complex business problems.

The book is divided into seven parts and three appendices. In Part One, "Orientation," the principles of basic computer programming are elucidated by means of a hypothetical computer which embodies an instruction repertoire of several existing machines. Various numerical and alphanumerical coding systems for storing data on punched cards, punched paper tapes, and magnetic tapes are also discussed here.

Part Two, "Automatic Equipment," deals with input-output hardware, storage devices, arithmetic and control units. The section concludes with a synopsis of the salient characteristics of approximately twenty computing systems: speed, storage, instruction repertoires, tapes, and peripheral equipment.

Advanced programming techniques and systems provide the subject matter of Part Three, "Programming and Processing Procedures." In this section the authors present a synthesis of the pros and cons of automatic programming and integrated data processing, two important and topical subjects.

The role of the data-processing unit in "management information systems" is the theme of Part Four, "Principles of Processing Systems." Several methods are suggested for selecting from a welter of available facts the pertinent information for effective executive decision-making. The reporting-by-exception principle is described in detail. Since the efficacy of the final system design is inextricably related to economic considerations, the authors analyze the major factors for determining the cost of obtaining and processing data, and explore the concept of the "value" of information in relation to its cost. The last chapter in Part Four outlines the broad principles underlying systems analysis and design.

Factors that affect the organizational structure of data processing are subject to examination in Part Five, "Systems Design." In particular, considerable attention is devoted to problems associated with centralized data processing and decentralized

management control. General tools for systems analysis and specific data-processing techniques are also described here.

Part Six, "Equipment Acquisition and Utilization," presents in a nontechnical manner a methodical approach for evaluating, selecting, installing, and implementing automatic data-processing systems for business problems. Considerable space is devoted to the preparation of feasibility studies, application studies, and equipment acquisition proposals. This is followed by a detailed discussion of the problems entailed in the installation of new equipment.

The concluding portion, Part Seven, "System Re-examination and Prospective Developments," touches on a variety of mathematical techniques for the solution of management problems and concludes with a discussion on anticipated future developments in automatic data processing.

Three appendices are:

I. History of Computation and Data-Processing Devices
II. Questions and Problems
III. Glossary of Automatic Data-Processing Terminology

Although the treatment of the basic principles of computer programming illuminates many of the complex and important aspects of business data processing, the authors give little heed to the practical requirements for large-scale production system runs. Such concepts and techniques as rerun procedures, interior tape labels, alternation of servos, and programming methods for effective utilization of buffering are not even mentioned, while the subjects of editing, flow charting of instruction routines, and sorting techniques for large tape files are glossed over. But on the whole, the informative and lucid presentation of the general principles of automatic data processing from the standpoint of business systems will provide management personnel with a short, intensive, and enlightened education on electronic computers and their impact on business data processing.

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61[Z].—Anthony G. Oettinger, *Automatic Language Translation*, Harvard University Press, Cambridge, 1960, xix + 380 p., 24 cm. Price $10.00.

Automatic Language Translation by Anthony G. Oettinger is the eighth in a series of Harvard Monographs in Applied Science. "These monographs are devoted primarily to reports of University research in the applied physical sciences, with special emphasis on topics that involve intellectual borrowing among the academic disciplines." Professor Oettinger's monograph is devoted to the lexical and technical aspects of automatic language translation, with particular emphasis on Russian-to-English translation.

The contents of this work can quickly be conveyed by the titles of its chapters. Chapter 1, "Automatic Information-Processing Machines," discusses the organization, elements of programming, and the characteristics of information-storage media. Chapter 2, "The Structure of Signs," differentiates the notions of use, mention, and representation of signs; mathematical transformations; and mathematical models. Chapter 3, "Flow Charts and Automatic Coding," treats the use of flow

charts, addressing, algorithms, and programs. Chapter 4, "The Problem of Translation," is a general discussion of the possibilities and types of automatic translation, grammar, and interlingual correspondence, and syntactic and semantic problems. Chapter 5, "Entry Keys for an Automatic Dictionary," discusses inflection, paradigms, affixes, stems, and inflection algorithms for Russian. Chapter 6, "Morphological and Functional Classification of Russian Words," consists of a detailed account of Oettinger's morphological and functional system for Russian words; nominal forms, adjectival forms, and verbal forms; and an appendix that gives synoptic classification tables. Chapter 7, "Dictionary Compilation," describes the structure of the Harvard automatic dictionary by giving the structure of items and files, methods of detecting and correcting mistakes in transcription and classification, and English correspondence and grammatical codes. Chapter 8, "Dictionary Operation," describes the function of the Harvard automatic dictionary, lookup procedures, and word-by-word translation; and is followed by an appendix that presents an edited trot, the transcription of the edited trot, and an example of conventional translation. Chapter 9, "Problems in Dictionary Compilation and Operation," discusses the problems of paradigm homography, stem homography, "short" words, and a detection and correction of mistakes in dictionary compilation. Chapter 10, "From Automatic Dictionary to Automatic Translator," presents the author's views on how the Harvard automatic dictionary might lead to a complete system of automatic translation.

Since this is the first book published in America devoted to automatic translation of languages, it is a landmark. Several cautions should be mentioned, however, for those who are not familiar with the state of progress in machine translation. First, this book is not a work devoted to the general problem of translating one natural language to another. It is highly specialized, since it treats only the Russian-to-English translation problem. Second, much of the book is devoted to the very detailed description of the particular automatic dictionary compiled at Harvard University. This description does not permit conclusions to be drawn "automatically" about dictionary compilation at other machine translation research centers. Third, all detailed computer descriptions are in terms of the Sperry-Rand UNIVAC I computer, whereas almost all other machine translation programs in the United States are written for IBM 704 or 709 computers.

Nevertheless, Professor Oettinger is to be congratulated for presenting the first detailed, and scientifically accurate description of any machine translation project in the U.S., if not in the world. As such, this book will be of interest to computer scientists, mathematicians, linguists, and to others interested in acquiring knowledge about this important subfield of modern linguistic analysis. As this reviewer has often emphasized, the gains to be made in linguistic analysis will overshadow those which have been made in numerical analysis.

H. P. EDMUNDSON

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TABLE ERRATA

299.—Mme. Jacqueline Heurtaux, "Tables de polynômes d'interpolation avec seulement deux abscisses distinctes," *Chiffres*, 1re Année, Paris, March 1958, p. 25–34.

		for	*read*
p. 31, Q_6^3,	$x = 0.15$	0.97338 82250	0.97338 81250

H. E. Salzer

Convair-Astronautics
San Diego 12, California

300.—H. Takeyamà, "Expressions for interpolation and numerical integration of high accuracy," *Tohoku Univ. Technol. Reports*, v. XXIII, 1958, p. 47–70.

On p. 69, corresponding to $u = 0.04$, the value of U_0' should read 0.039 . . . instead of 9.039 . . . ; and corresponding to $u = 0.34$, the value of U_2 should read 0.72203 53338 6336 instead of 0.72203 58338 6336.

H. E. Salzer

CORRIGENDA

C. W. Dunnett & R. A. Lamm, "Some tables of the multivariate normal probability integral with correlation coefficients $\frac{1}{3}$," *Math. Comp.*, Review 50, v. 14, 1960, p. 290.

In the expression given for the probability integral of the multivariate normal distribution in n dimensions the upper limit of the innermost integral should read x_n instead of x_m, and the denominator $(1 - \rho)\frac{(n-1)}{2}$ should be replaced by $(1 - \rho)^{\frac{(n-1)}{2}}$.

In the following line of the text

$$\textit{for} \quad F_{n,\rho}(x_1, \cdots, k_n), \quad \textit{read} \quad F_{n,\rho}(x_1, \cdots, x_n).$$

F. R. Gantmacher, *Applications of the Theory of Matrices*, *Math. Comp.* Review 43, v. 14, 1960, p. 284–285.

This book is a translation and revision of the second volume of Gantmacher's *Theory of Matrices* that was carried out by three people; namely, J. L. Brenner (named as the sole translator in the review under discussion), Mr. S. Evanusa and Prof. D. W. Bushaw.

Murlan S. Corrington, "Applications of the complex exponential integral," *Math. Comp.*, v. 15, 1961, p. 1–6.

On p. 2, eq. (11c) should read $Si(-x - iy) = -Si(x + iy)$ in place of $Si(-x - iy) = Si(x + iy)$.

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CLASSIFICATION OF REVIEWS

A. Arithmetical Tables, Mathematical Constants
B. Powers
C. Logarithms
D. Circular Functions
E. Hyperbolic and Exponential Functions
F. Theory of Numbers
G. Higher Algebra
H. Numerical Solution of Equations
I. Finite Differences, Interpolation
J. Summation of Series
K. Statistics
L. Higher Mathematical Functions
M. Integrals
N. Interest and Investment
O. Actuarial Science
P. Engineering
Q. Astronomy
R. Geodesy
S. Physics, Geophysics, Crystallography
T. Chemistry
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Z. Calculating Machines and Mechanical Computation

Mathematics of Computation

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APRIL 1961

Published Quarterly by the

National Academy of Sciences–National Research Council